Stability of contact discontinuity for the Navier-Stokes-Poisson system with free boundary

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Abstract

This paper is concerned with the study of the nonlinear stability of the contact discontinuity of the Navier-Stokes-Poisson system with free boundary in the case where the electron background density satisfies an analogue of the Boltzmann relation. We especially allow that the electric potential can take distinct constant states at boundary. On account of the quasineutral assumption, we first construct a viscous contact wave through the quasineutral Euler equations, and then prove that such a non-trivial profile is time-asymptotically stable under small perturbations for the corresponding initial boundary value problem of the Navier-Stokes-Poisson system. The analysis is based on the techniques developed in [11] and an elementary L^2 energy method.

Key words. viscous contact discontinuity, quasineutral Euler equations, stability, free boundary.

AMS subject classifications. 35B35, 35Q35, 82D10.

Contents

1 Introduction

1.1 The problem

The dynamics of the charged particles in the collisional dusty plasma can be described by the Navier-Stokes-Poisson (denoted as NSP in the sequel) system [15]. The one-dimensional NSP system in the Eulerian coordinates takes the form of

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \rho \partial_x \phi + \mu \partial_x^2 u, \\ \partial_t W + \partial_x(W u + p u) = \rho u \partial_x \phi + \mu \partial_x(u \partial_x u) + \kappa \partial_x^2 \theta, \\ \partial_x^2 \phi = \rho - \rho_e(\phi). \end{cases} \tag{1.1}$$

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The unknown functions ρ, u and θ stand for the density, velocity and absolute temperature of ions, respectively, while $\mu > 0$ is the viscosity coefficient and $\kappa > 0$ is the heat conductivity coefficient. W stands for the total energy of the ions, taking the following form:

$$W = \frac{\rho u^2}{2} + \frac{p}{\gamma - 1},$$

where $\gamma > 1$ is the adiabatic exponent. p is the pressure which is given by

$$p = R\rho\theta = A\rho^\gamma e^{\frac{\gamma-1}{R}S},$$

where S is the entropy and A, R are both positive constants. The self-consistent electric potential $\phi = \phi(x,t)$ is induced by the total charges through the Poisson equation. The density $\rho_e = \rho_e(\phi)$ of electrons in (1.1) depends only on the potential in the sense of an analogue of the so-called Boltzmann relation, cf. [5, 20]. Specifically, through the paper we suppose that

($\mathcal{A}$) $\rho_e(\phi) : (\phi_m, \phi_M) \to (\rho_m, \rho_M)$ is a smooth function with

$$\rho_m = \inf_{\phi_m < \phi < \phi_M} \rho_e(\phi), \quad \rho_M = \sup_{\phi_m < \phi < \phi_M} \rho_e(\phi),$$

satisfying the following two assumptions:

($\mathcal{A}_1$) $\rho_e(0) = 1$ with $0 \in (\phi_m, \phi_M)$;
($\mathcal{A}_2$) $\rho_e(\phi) > 0$, $\rho_e'(\phi) < 0$ for each $\phi \in (\phi_m, \phi_M)$.

The assumption ($\mathcal{A}_1$) just means that the electron density has been normalized to be unit when the potential is zero, since the electric potential in (1.1) can be up to an arbitrary constant. The sign of the first derivative of the function $\rho_e(\phi)$ in the assumption ($\mathcal{A}_2$) plays a crucial role in our analysis, it is to be further clarified later on, see (1.16), etc.

An important example satisfying ($\mathcal{A}$) can be given as

$$\rho_e(\phi) = \left[1 - \frac{\gamma_e - 1}{\gamma_e}\frac{\phi}{A_e}\right]^{\frac{1}{\gamma_e - 1}}, \quad \phi_m = -\infty, \quad \phi_M = \frac{\gamma_e}{\gamma_e - 1}A_e, \tag{1.2}$$

with $\gamma_e \geq 1$ and $A_e > 0$ being constants. Note that $\rho_e(\phi) \to e^{-\frac{\phi}{A_e}}$ and $\phi_M \to +\infty$ as $\gamma_e \to 1^+$, which corresponds to the classical Boltzmann relation. In fact, (1.2) can be formally deduced from the momentum equation of the isentropic Euler-Poisson system for the fluid of electrons with the adiabatic exponent γ_e under the zero-limit of electron mass, namely, $\partial_x\left(A_e\rho_e^{\gamma_e}\right) = -\rho_e\partial_x\phi$.

In this paper, we consider the system (1.1) in the part $+\infty > x \geq x(t)$, where $x = x(t)$ is a free boundary with the following dynamical boundary conditions

$$\frac{dx(t)}{dt} = u(x(t),t), \quad x(0) = 0, \quad (p - \mu\partial_x u)\,|_{x=x(t)} = p_-, \quad \theta(x(t),t) = \theta_-, \quad \phi(x(t),t) = \phi_-. \tag{1.3}$$

We also assume ϕ satisfies the boundary condition at far field:

$$\lim_{x \to +\infty} \phi(x,t) = \phi_+. \tag{1.4}$$

The initial data is given by

$$(\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x), \quad \lim_{x \to +\infty}(\rho_0, u_0, \theta_0)(x) = (\rho_+, u_+, \theta_+). \tag{1.5}$$

Here $\rho_+ > 0$, $\theta_\pm > 0$, $p_- > 0$, u_+ and $\phi_\pm$ are assumed to be constant states. Also, $\rho_0(x) > 0$ is supposed, so that the ions flow has no vacuum state. In addition, we of course assume $\theta_0(x)$ satisfies the compatibility condition and ϕ satisfies the the quasineutral condition at far field, i.e.

$$\theta_0(0) = \theta_-, \quad \rho_e(\phi_+) = \rho_+. \tag{1.6}$$

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Our main purpose concerns the large time behavior of solutions to (1.1), (1.3), (1.4) and (1.5), to explore this, it is more convenient to use the Lagrangian coordinates. That is, consider the coordinate transformation:

$$x \Rightarrow \int_{x(t)}^{x} \rho(y,t)dy, \quad t \Rightarrow t.$$

We still denote the Lagrangian coordinates by (x,t) for simplicity of notation. Noticing that

$$\int_{x(t)}^{x} \rho(y,t)dy \to +\infty, \text{ as } x \to +\infty,$$

one sees that (1.1), (1.3), (1.4) and (1.5) can be transformed as the problem with fixed boundary in the form of

$$\begin{cases} \partial_t v - \partial_x u = 0, \quad x > 0, \ t > 0, \\ \partial_t u + \partial_x p = \dfrac{\partial_x \phi}{v} + \mu \partial_x \left(\dfrac{\partial_x u}{v} \right), \quad x > 0, \ t > 0, \\ \dfrac{R}{\gamma - 1} \partial_t \theta + p \partial_x u = \mu \dfrac{(\partial_x u)^2}{v} + \kappa \partial_x \left(\dfrac{\partial_x \theta}{v} \right), \quad x > 0, \ t > 0, \\ \partial_x \left(\dfrac{\partial_x \phi}{v} \right) = 1 - v \rho_e(\phi), \quad x > 0, \ t > 0, \end{cases} \tag{1.7}$$

with boundary condition

$$\theta(0,t) = \theta_-, \ \left(p - \mu \dfrac{\partial_x u}{v} \right)(0,t) = p_-, \ \phi(0,t) = \phi_-, \ \lim_{x \to +\infty} \phi(x,t) = \phi_+, \ t \geq 0, \tag{1.8}$$

and the initial data

$$(v,u,\theta)(x,0) = (v_0, u_0, \theta_0)(x), \quad x \geq 0, \quad \lim_{x \to +\infty} (v_0, u_0, \theta_0)(x) = (v_+, u_+, \theta_+). \tag{1.9}$$

Here $v = 1/\rho$ stands for the specific volume. Moreover,

$$\theta_0(0) = \theta_- \text{ and } v_+ = \frac{1}{\rho_e(\phi_+)}$$

hold according to (1.6).

1.2   Quasineutral Euler equations and contact waves

In order to study the large time behavior of the solution $[v(x,t), u(x,t), \theta(x,t), \phi(x,t)]$ to the initial boundary value problem (1.7), (1.8) and (1.9), we expect that $[v(x,t), u(x,t), \theta(x,t), \phi(x,t)]$ tends time-asymptotically to viscous contact wave to the Riemann problem on the quasineutral Euler system

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = \dfrac{\partial_x \phi}{v}, \\ \dfrac{R}{\gamma - 1} \partial_t \theta + p \partial_x u = 0, \\ 1/v = \rho_e(\phi), \end{cases} \tag{1.10}$$

with Riemann initial data given by

$$[v,u,\theta](x,0) = \begin{cases} [v_-, u_-, \theta_-], & x < 0, \\ [v_+, u_+, \theta_+], & x > 0. \end{cases} \tag{1.11}$$

According to [6, 43], one sees that the Riemann problem (1.10) and (1.11) admits a contact discontinuity solution

$$\left[v^{CD}, u^{CD}, \theta^{CD}, \phi^{CD} \right](x,t) = \begin{cases} [v_-, u_-, \theta_-, \phi_-], & x < 0, \\ [v_+, u_+, \theta_+, \phi_+], & x > 0, \end{cases} \tag{1.12}$$

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on the condition that

$$u_- = u_+, \quad p_- \overset{\text{def}}{=} p(v_-, \theta_-) = p_+ + p^\phi(v_+) - p^\phi(v_-), \tag{1.13}$$

where

$$p_+ = p(v_+, \theta_+), \ \phi_\pm = \rho_e^{-1}(1/v_\pm) \text{ and } p^\phi = p^\phi(v) = \int^v \frac{1}{\varrho^3 \rho_e'(\rho_e^{-1}(\frac{1}{\varrho}))} d\varrho.$$

On the other hand, due to the dissipation effect of the NSP system (1.7), a viscous contact wave $\left[v^{cd}, u^{cd}, \theta^{cd}, \phi^{cd}\right]$ corresponding to the contact discontinuity $\left[v^{CD}, u^{CD}, \theta^{CD}, \phi^{CD}\right]$ defined as (1.12) can be constructed as follows. We first denote $p^{cd} = p^{cd}(v^{cd}, \theta^{cd}) = \frac{R\theta^{cd}}{v^{cd}}$. Since the quasineutral pressure $p^{cd} + p^\phi$ for the profile $\left[v^{cd}, u^{cd}, \theta^{cd}, \phi^{cd}\right]$ is expected to be almost constant, we set

$$p_- = p^{cd} + \int_{v_-}^{v^{cd}} \frac{1}{\varrho^3 \rho_e'(\rho_e^{-1}(\frac{1}{\varrho}))} d\varrho. \tag{1.14}$$

Noticing that $\frac{\partial p^{cd}}{\partial v^{cd}} < 0$ and $\rho_e'(\cdot) < 0$, from which and (1.14) and the implicit function theorem, we see that there exists a differentiable function $f(\theta^{cd})$ such that

$$v^{cd} = f(\theta^{cd}), \ v_\pm = f(\theta_\pm), \tag{1.15}$$

provided that $|\theta_+ - \theta_-|$ is suitably small. Furthermore, by a direct calculation, it follows that

$$f'(\theta^{cd}) = \frac{R}{p^{cd} - \frac{1}{(v^{cd})^2 \rho_e'(\phi^{cd})}} > 0. \tag{1.16}$$

We now rewrite the leading part of $(1.7)_3$ (the third equation of (1.7)) as

$$\frac{R}{\gamma - 1} \partial_t \theta^{cd} + p^{cd} \partial_x u^{cd} = \kappa \partial_x \left(\frac{\partial_x \theta^{cd}}{v^{cd}}\right). \tag{1.17}$$

With (1.15) and (1.17) in hand, we further conjecture that $\left[v^{cd}, u^{cd}, \theta^{cd}\right]$ satisfies

$$\begin{cases} \partial_t v^{cd} - \partial_x u^{cd} = 0, \ v^{cd} = f(\theta^{cd}), \\ \dfrac{R}{\gamma - 1} \partial_t \theta^{cd} + p^{cd} \partial_x u^{cd} = \kappa \partial_x \left(\dfrac{\partial_x \theta^{cd}}{v^{cd}}\right), \\ \theta^{cd}(0, t) = \theta_-, \ \theta^{cd}(+\infty, t) = \theta_+, \ v^{cd}(0, t) = v_-, \ v^{cd}(+\infty, t) = v_+. \end{cases} \tag{1.18}$$

By virtue of (1.18), we obtain a nonlinear diffusion equation as follows:

$$\partial_t \theta^{cd} = \frac{\kappa}{g(\theta^{cd})} \partial_x \left(\frac{\partial_x \theta^{cd}}{f(\theta^{cd})}\right), \quad \theta^{cd}(0, t) = \theta_-, \quad \theta^{cd}(+\infty, t) = \theta_+, \tag{1.19}$$

where $g(\theta^{cd}) = \frac{R}{\gamma - 1} + p^{cd} f'(\theta^{cd}) > 0$. Applying the same argument as in [1], one sees that (1.19) admits a unique self similarity solution $\theta^{cd}(\xi)$, $\xi = \frac{x}{\sqrt{1+t}}$. Additionally, it turns out that θ^{cd} is a monotone function, increasing if $\theta_+ > \theta_-$ and decreasing if $\theta_+ < \theta_-$, and more importantly, one can show that there exists some positive constant $\overline{\delta}$, such that for $\delta = |\theta_+ - \theta_-| \leq \overline{\delta}$, θ^{cd} satisfies

$$(1 + t) \left|\partial_x^2 \theta^{cd}\right| + (1 + t)^{\frac{1}{2}} \left|\partial_x \theta^{cd}\right| + \left|\theta^{cd} - \theta_\pm\right| \leq C\delta e^{-\frac{c_1 x^2}{1+t}}, \quad \text{as} \ x \to +\infty, \tag{1.20}$$

where c_1 is some positive constant. After θ^{cd} and v^{cd} are obtained, we now define $\left[u^{cd}, \phi^{cd}\right]$ as follows

$$\begin{cases} \phi^{cd} = \rho_e^{-1}(1/v^{cd}), \\ u^{cd} = u_+ - \kappa \int_x^{+\infty} \frac{f'(\theta^{cd})}{g(\theta^{cd})} \partial_x \left(\frac{\partial_x \theta^{cd}}{f(\theta^{cd})}\right) dx \\ \quad = u_+ + \frac{\kappa f'(\theta^{cd})}{g(\theta^{cd}) f(\theta^{cd})} \partial_x \theta^{cd} + \kappa \int_x^{+\infty} \frac{(\partial_x \theta^{cd})^2}{f(\theta^{cd})} \left(\frac{f'}{g}\right)'(\theta^{cd}) dx, \\ \phi^{cd}(0, t) = \phi_-, \ \phi^{cd}(+\infty, t) = \phi_+, \ u^{cd}(+\infty, t) = u_+. \end{cases} \tag{1.21}$$

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It should be noted that $\phi_\pm = \rho_e^{-1}(1/v_\pm)$, and $u^{cd}(0,t)$ may not equal to u_+.

In view of (1.12), (1.18), (1.20) and (1.21), it is straightforward to compute that $\left[v^{cd}, u^{cd}, \theta^{cd}, \phi^{cd}\right]$ satisfies

$$\left\| \left[v^{cd} - v^{CD}, u^{cd} - u^{CD}, \theta^{cd} - \theta^{CD}, \phi^{cd} - \phi^{CD}\right] \right\|_{L^p(\mathbb{R}_+)} = O\left(\kappa^{\frac{1}{2p}}\right)(1+t)^{\frac{1}{2p}}, \quad p \geq 1,$$

which implies the viscous contact wave $\left[v^{cd}, u^{cd}, \theta^{cd}, \phi^{cd}\right](x,t)$ constructed in (1.18) and (1.21) approximates the contact discontinuity solution $\left[v^{CD}, u^{CD}, \theta^{CD}, \phi^{CD}\right]$ to the quasineutral Euler system (1.10) in L^p norm, $p \geq 1$ on any finite time interval as the heat conductivity coefficients κ tends to zero. Moreover, we see that the viscous contact wave $\left[v^{cd}, u^{cd}, \theta^{cd}, \phi^{cd}\right](x,t)$ solves the Navier-Stokes-Poisson system (1.7) time asymptotically, that is,

$$\begin{cases} \partial_t v^{cd} - \partial_x u^{cd} = 0, \\[2mm] \partial_t u^{cd} + \partial_x p^{cd} = \dfrac{\partial_x \phi^{cd}}{v^{cd}} + \mu \partial_x \left(\dfrac{\partial_x u^{cd}}{v^{cd}}\right) + \mathcal{R}_1, \\[2mm] \dfrac{R}{\gamma - 1} \partial_t \theta^{cd} + p^{cd} \partial_x u^{cd} = \mu \dfrac{(\partial_x u^{cd})^2}{v^{cd}} + \kappa \partial_x \left(\dfrac{\partial_x \theta^{cd}}{v^{cd}}\right) + \mathcal{R}_2, \\[2mm] \partial_x \left(\dfrac{\partial_x \phi^{cd}}{v^{cd}}\right) = 1 - v^{cd} \rho_e(\phi^{cd}) + \mathcal{R}_3, \end{cases}$$

where

$$\mathcal{R}_1 = \partial_t \left(\frac{\kappa f'(\theta^{cd})}{g(\theta^{cd}) f(\theta^{cd})} \partial_x \theta^{cd} + \int_x^{+\infty} \frac{\kappa (\partial_x \theta^{cd})^2}{f(\theta^{cd})} \left(\frac{f'}{g}\right)' (\theta^{cd}) dx \right) - \mu \partial_x \partial_t \left[\ln \left(f(\theta^{cd})\right)\right]$$

$$= O(\delta)(1+t)^{-\frac{3}{2}} e^{-\frac{c_1 x^2}{1+t}}, \quad \text{as} \quad x \to +\infty,$$

$$\mathcal{R}_2 = -\mu \frac{\left(f'(\theta^{cd}) \partial_t \theta^{cd}\right)^2}{f(\theta^{cd})} = O(\delta)(1+t)^{-2} e^{-\frac{c_1 x^2}{1+t}}, \quad \text{as} \quad x \to +\infty,$$

and

$$\mathcal{R}_3 = \partial_x \left(\frac{\partial_x \phi^{cd}}{v^{cd}}\right) = O(\delta)(1+t)^{-1} e^{-\frac{c_1 x^2}{1+t}}, \quad \text{as} \quad x \to +\infty.$$

1.3 Main results

Now we are in a position to state our main results.

Theorem 1.1. *For any given $[v_+, u_+, \theta_+, p_-]$ with $v_+ > 0$ and $\theta_+ > 0$, suppose that $[v_-, u_-, \theta_-]$ satisfies (1.13), $\phi_\pm = \rho_e^{-1}(v_\pm)$ with $\phi_\pm \in (\phi_m, \phi_M)$, and the function $\rho_e(\cdot)$ satisfies the assumption $(\mathcal{A})$. Let $\left[v^{cd}, u^{cd}, \theta^{cd}, \phi^{cd}\right](x,t)$ be the viscous contact wave defined in (1.18) and (1.21) with strength $\delta = |\theta_+ - \theta_-|$. There exist positive constants $\epsilon_0 > 0$ and $C_0 > 0$, such that if $\left[v_0(x) - v^{cd}(x,0), u_0(x) - u^{cd}(x,0)\right] \in H^1$, $\left[\theta_0(x) - \theta^{cd}(x,0)\right] \in H_0^1$ and*

$$\left\| \left[v_0(x) - v^{cd}(x,0), u_0(x) - u^{cd}(x,0), \theta_0(x) - \theta^{cd}(x,0)\right] \right\|_{H^1} + \delta \leq \epsilon_0,$$

then the initial boundary value problem (1.7), (1.8) and (1.9) admits a unique global solution $[v, u, \theta, \phi](x,t)$ satisfying $\left[v - v^{cd}, u - u^{cd}\right] \in C(0, +\infty; H^1)$, $\left[\theta(x) - \theta^{cd}, \phi - \phi^{cd}\right] \in C(0, +\infty; H_0^1)$ and

$$\sup_{t \geq 0} \left\| \left[v - v^{cd}, u - u^{cd}, \theta - \theta^{cd}, \phi - \phi^{cd}\right] \right\|_{H^1} \leq C_0 \epsilon_0^{2/3}. \tag{1.22}$$

Moreover, it holds that

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}_+} \left| \left[v - v^{cd}, u - u^{cd}, \theta - \theta^{cd}, \phi - \phi^{cd}\right] \right| = 0. \tag{1.23}$$

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From a physical point of view, the motion of the ion-dust plasma (cf. [32, 15]), the self-gravitational viscous gaseous stars (cf. [3]) and the charged particles in semiconductor devices (cf. [37]) can be governed by the NSP system. On the other hand, the NSP system at the fluid level can be justified by taking the hydrodynamical limit of the Vlasov-type Boltzmann equation by the Chapman-Enskog expansion, cf. [4, 17, 18, 19]. In recent years, there have been a great number of mathematical studies of the NSP system. In what follows, we only mention some of them related to our interest. Ducomet [14] obtained the existence of nontrivial stationary solutions with compact support and proved the dynamical stability related to a free-boundary value problem for the three-dimensional NSP system in the case that the background profile is vacuum. Donatelli [8] established the global existence of weak solutions to the Cauchy problem with large initial data. Recently, Ding-Wen-Yao-Zhu [7] proved the global existence of weak solutions to the one dimensional isentropic NSP system with density-dependent viscosity and free boundary. Donatelli-Marcati [9] studied the quasineutral limit by using some dispersive estimates of Strichartz type. We point out that some nonexistence result of global weak solutions was also obtained in Chae [2]. Zhang-Fang [47] studied the large-time behavior of the spherically symmetric NSP system with degenerate viscosity coefficients and with vacuum in three dimensions. Jang-Tice [28] investigated the linear and nonlinear dynamical instability for the Lane-Emden solutions of the NSP system in three dimensions under some condition on the adiabatic exponent. Tan-Yang-Zhao-Zou [44] established the global strong solution to the one-dimensional non-isentropic NSP system with large data for density-dependent viscosity. In the case when the background profile is strictly positive, the global existence and convergence rates for the three-dimensional NSP system around a non-vacuum constant state were studied by Li-Matsumura-Zhang [33], Zhang-Li-Zhu [46] and Hsiao-Li [21] through carrying out the spectrum analysis. We point out that Duan [10] also used the method of Green's function to obtain the large time behaviors of the more complex Navier-Stokes-Maxwell system.

Another interesting and challenging problem is to study the stability of the NSP system on half space, to the best of our knowledge, there are very few results in this line. Duan-Yang [13] recently proved the stability of rarefaction wave and boundary layer for outflow problem on the two-fluid NSP system. The convergence rate of corresponding solutions toward the stationary solution was obtained in Zhou-Li [48]. We remark that due to the techniques of the proof, it was assumed in [13] that all physical parameters in the model must be unit, which is obviously impractical since ions and electrons generally have different masses and temperatures. One important point used in [13] is that the large-time behavior of the electric potential is trivial and hence the two fluids indeed have the same asymptotic profiles which are constructed from the Navier-Stokes equations without any force instead of the quasineutral system. Duan-Liu [11] then improved the results of [13] in the sense that all physical constants appearing in the model can be taken in a general way, and the large-time profile of the electric potential is nontrivial on the basis of the quasineutral assumption. For the investigations in the stability of the rarefaction wave of the related models, see also [12] for the study of the more complicated Vlasov-Poisson-Boltzmann system with more general background profile.

When there is no self-consistent force, the NSP system reduces to the well-known Navier-Stokes equations. It is known that there have been extensive investigations on the stability of wave patterns, namely, shock wave, rarefaction wave, contact discontinuity and their compositions, in the context of gas dynamical equations and related kinetic equations. Among them, we only mention [16, 23, 27, 29, 30, 31, 34, 35, 36, 38, 40, 41, 39, 42, 45] and reference therein. Moreover, we would also point out some previous works only related to the current work. Huang-Mastumura-Shi [24] proved the stability of contact discontinuity of compressible Navier-Stokes equations with free boundary for the ideal polytropic gas through the construction of viscous contact wave profiles, the key observation in [24] is that the asymptotic profile of the temperature θ satisfies a nonlinear diffusion equation, which can be solved by the technique developed in [1, 22], and later on Huang-Mastumura-Xin [25] and Huang-Li-Mastumura [23] established the stability of the contact waves of the Cauchy problem. Recently Huang-Wang-Zhai [26] extended the results in [24] to the general gas, however, for the Cauchy problem, it still remains an interesting open problem to generalize the results in [25, 23] for the general gas.

In this paper, we intend to study the stability of the contact wave of the NSP system (1.1) with free boundary. Motivated by [11] and [24], we first construct the nontrivial asymptotic profiles of the quasineural Euler equations, it should be noted that the background density $\rho_e(\phi)$ satisfying assumption $(\mathcal{A})$ allows that the asymptotic profile of the electrical potential can be distinct at the boundary. Then we perform the elementary energy estimates to the perturbative equations to obtain the global existence and

the large time behaviors. Compared to the classical Navier-Stokes system without any force, the main difficulty in the proof for the NSP system is to treat the estimates on the terms caused by the potential function ϕ. Precisely, the delicate term $\left(\frac{\partial_x \phi}{v} - \frac{\partial_x \phi^{cd}}{v^{cd}}\right)\psi$ can not be directly controlled, as in [11], the key point to overcome the difficulty is to use the good dissipative property from the Poisson equation by expanding $\rho_e(\phi)$ around the asymptotic profile up to the third-order. In addition, it is shown [11] that the sign of the first derivative of the rarefaction profile of the velocity and the good time decay properties of the smooth rarefaction profiles are important to the *a priori* estimate. Thus compared with [11] in which the stability of the rarefaction wave of the NSP system is proved, a new difficulty will arise, that is, the critical term $\int_0^T \int_{\mathbb{R}_+} \psi^2 (\partial_x \theta^{cd})^2 dxdt$ is beyond control, unlike that of [23], we need to pay extra effort to take care of the terms involving the self-consistent force, and it can be seen that the assumption $(\mathcal{A}_2)$ plays an essential role to obtain the desired estimates, see Lemma 4.3 for the details.

The rest of the paper is arranged as follows. In the main part Section 2, we give the *a priori* estimates on the solutions of the perturbative equations. The proof of Theorem 1.1 is concluded in Section 3. In the Appendix, we present the details that are left in the proofs of the previous sections for completeness of the paper.

Notations. Throughout this paper, we denote a generally large constant by C, which may vary from line to line. For two quantities a and b, $a \sim b$ means $\frac{1}{C}a \leq b \leq Ca$. $L^p = L_x^p(\mathbb{R}_+)$ $(1 \leqslant p \leqslant \infty)$ denotes the usual Lebesgue space on $\mathbb{R}_+ = [0, +\infty)$ with its norm $\| \cdot \|_{L^p}$, and for convenient, we write $\| \cdot \|_{L^2} = \| \cdot \|$. We also use H^k $(k \geq 0)$ to denote the usual Sobolev space with respect to x variable on $\mathbb{R}_+$. $C([0, T]; H^k)(k \geq 0)$ denotes the space of the continuous functions on the interval $[0, T]$ with values in H^k. We use $(\cdot, \cdot)$ to denote the inner product over the Hilbert space L^2. $[f_1, f_2] \in H^1$ means $f_1 \in H^1$ and $f_2 \in H^1$, and so on so forth.

2   The a priori estimates

In order to study the stability of contact wave of the initial boundary value problem (1.7), (1.8) and (1.9), that is, to prove Theorem 1.1, we first define the perturbation as

$$[\varphi, \psi, \zeta, \sigma](x, t) = \left[v - v^{cd}, u - u^{cd}, \theta - \theta^{cd}, \phi - \phi^{cd}\right](x, t).$$

Then $[\varphi, \psi, \zeta, \sigma](x, t)$ satisfies

$$\partial_t \varphi - \partial_x \psi = 0, \tag{2.1}$$

$$\partial_t \psi + \partial_x p - \partial_x p^{cd} = \left(\frac{\partial_x \phi}{v} - \frac{\partial_x \phi^{cd}}{v^{cd}}\right) + \mu \partial_x \left(\frac{\partial_x \psi}{v}\right) + F, \tag{2.2}$$

$$\frac{R}{\gamma - 1} \partial_t \zeta + p \partial_x u - p^{cd} \partial_x u^{cd} = \kappa \partial_x \left(\frac{\partial_x \theta}{v} - \frac{\partial_x \theta^{cd}}{v^{cd}}\right) + G, \tag{2.3}$$

$$v^{cd} \partial_x \left(\frac{\partial_x \sigma}{v}\right) = -\varphi + v \left[1 - v^{cd} \rho_e \left(\sigma + \phi^{cd}\right)\right] - v^{cd} \partial_x \left(\frac{\partial_x \phi^{cd}}{v}\right), \tag{2.4}$$

$$\left(p(v, \theta) - \mu \frac{\partial_x u}{v}\right)(0, t) = p_-, \ \zeta(0, t) = \sigma(0, t) = \sigma(+\infty, t) = 0, \tag{2.5}$$

$$[\varphi, \psi, \zeta](x, 0) = [\varphi_0, \psi_0, \zeta_0](x)$$
$$= \left[v_0(x) - v^{cd}(x, 0), u_0(x) - u^{cd}(x, 0), \theta_0(x) - \theta^{cd}(x, 0)\right], \tag{2.6}$$

where $x \geq 0$, $t \geq 0$, $F = -\partial_t u^{cd} + \mu \partial_x(\frac{\partial_x u^{cd}}{v})$ and $G = \mu \frac{(\partial_x u)^2}{v}$. We note that the structural identity (2.4) will be of extremal importance for the later proof.

The local existence of (1.7), (1.8) and (1.9) can be established by the standard iteration argument cf. [24] and hence will be skipped in the paper. To obtain the global existence part of Theorem 1.1, it suffices to prove the following *a priori* estimates. For results in this direction, we have

Proposition 2.1. *Assume all the conditions listed in Theorem 1.1 hold. Let $[\varphi, \psi, \zeta, \sigma]$ be a solution to the initial boundary value problem (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) on $0 \leq t \leq T$ for some*

positive constant T. *There are constants* $\delta > 0$, $\epsilon_0 > 0$ *and* $C > 0$, *such that if* $[\varphi, \psi] \in C(0, T; H^1)$, $[\zeta, \sigma] \in C(0, T; H_0^1)$ *and*

$$\sup_{0 \le t \le T} \|[\varphi, \psi, \zeta, \sigma](t)\|_{H^1} + \delta \le \epsilon_0, \tag{2.7}$$

then the solution $[\varphi, \psi, \zeta, \sigma](x, t)$ *satisfies*

$$\sup_{0 \le t \le T} \|[\varphi, \psi, \zeta, \sigma](t)\|_{H^1}^2 + \int_0^T \|\partial_x \varphi\|^2 + \|\partial_x [\psi, \zeta, \sigma]\|_{H^1}^2 dt \tag{2.8}$$

$$\le C\delta + C \|[\varphi_0, \psi_0, \zeta_0]\|_{H^1}^{4/3}.$$

Proof. We divide it by the following three steps.

Step 1. The zero-order energy estimates.

Multiplying (2.1), (2.2) and (2.3) by $-R\theta^{cd} \left(\frac{1}{v} - \frac{1}{v^{cd}} \right)$, ψ and $\zeta\theta^{-1}$, respectively, then taking the summation of the resulting equations, we obtain

$$\partial_t \left(\frac{1}{2}\psi^2 + R\theta^{cd}\Phi\left(\frac{v}{v^{cd}}\right) + \frac{R}{\gamma - 1}\theta^{cd}\Phi\left(\frac{\theta}{\theta^{cd}}\right) \right) + \mu\frac{(\partial_x \psi)^2}{v}$$

$$+ \frac{\kappa}{v\theta}(\partial_x \zeta)^2 + H_x + Q_1 + Q_2 = F\psi + \frac{\zeta}{\theta}G + \underbrace{\left(\frac{\partial_x \phi}{v} - \frac{\partial_x \phi^{cd}}{v^{cd}} \right)\psi}_{I_1}, \tag{2.9}$$

where

$$\Phi(s) = s - 1 - \ln s,$$

$$H = \left(p - p^{cd} \right)\psi - \mu\frac{\psi\partial_x \psi}{v} - \kappa\frac{\zeta}{\theta}\left(\frac{\partial_x \theta}{v} - \frac{\partial_x \theta^{cd}}{v^{cd}} \right),$$

$$Q_1 = -R\partial_t \theta^{cd}\Phi\left(\frac{v}{v^{cd}}\right) - p^{cd}\partial_t v^{cd}\left(2 - \frac{v}{v^{cd}} - \frac{v^{cd}}{v} \right) + \frac{R}{\gamma - 1}\partial_t \theta^{cd}\Phi\left(\frac{\theta^{cd}}{\theta}\right) + \frac{\zeta}{\theta}\left(p - p^{cd} \right)\partial_x u^{cd},$$

and

$$Q_2 = -\kappa\frac{\partial_x \theta}{\theta^2 v}\zeta\partial_x \zeta - \kappa\frac{\varphi\partial_x \zeta}{\theta v v^{cd}}\partial_x \theta^{cd} + \kappa\frac{\zeta\varphi\partial_x \theta}{\theta^2 v v^{cd}}\partial_x \theta^{cd}.$$

Let us now consider the most delicate term I_1 on the right hand side of (2.9). The key technique to handle I_1 is to use the good dissipative property of the Poisson equation by expanding $\rho_e(\sigma + \phi^{cd})$ around the asymptotic profile up to the third-order. Only in this way, we can observe some new cancelations and obtain the higher order nonlinear terms.

With the aid of (2.4) and (2.1), one has

$$I_1 = -\frac{\partial_x \psi\sigma}{v} + \frac{\psi\partial_x \varphi\sigma}{v^2} + \frac{\psi\partial_x v^{cd}\sigma}{v^2} + \frac{\psi\partial_x v^{cd}\varphi}{(v^{cd})^3 v\rho'_e(\phi^{cd})} + \partial_x\left(\frac{\sigma\psi}{v}\right)$$

$$= \underbrace{-\partial_t\left[-v^{cd}\partial_x\left(\frac{\partial_x \sigma}{v}\right) + v\left(1 - v^{cd}\rho_e(\sigma + \phi^{cd}) \right) - v^{cd}\partial_x\left(\frac{\partial_x \phi^{cd}}{v}\right) \right]\sigma v^{-1}}_{I_{1,1}}$$

$$\underbrace{+\partial_x\left[-v^{cd}\partial_x\left(\frac{\partial_x \sigma}{v}\right) + v\left(1 - v^{cd}\rho_e(\sigma + \phi^{cd}) \right) - v^{cd}\partial_x\left(\frac{\partial_x \phi^{cd}}{v}\right) \right]\psi\sigma v^{-2}}_{I_{1,2}} \tag{2.10}$$

$$\underbrace{+\left[-v^{cd}\partial_x\left(\frac{\partial_x \sigma}{v}\right) + v\left(1 - v^{cd}\rho_e(\sigma + \phi^{cd}) \right) - v^{cd}\partial_x\left(\frac{\partial_x \phi^{cd}}{v}\right) \right]\psi\partial_x v^{cd}(v^{cd})^{-3}[v\rho'_e(\phi^{cd})]^{-1}}_{I_{1,3}}$$

$$\underbrace{+\frac{\psi\partial_x v^{cd}\sigma}{v^2}}_{I_2} + \partial_x\left(\frac{\sigma\psi}{v}\right).$$

8

To deal with the lower order terms involving $1 - v^{cd}\rho_e(\sigma + \phi^{cd})$, we first get from the Taylor's formula with an integral remainder that

$$1 - v^{cd}\rho_e(\sigma + \phi^{cd}) = -v^{cd}\rho_e'(\phi^{cd})\sigma - \frac{v^{cd}\rho_e''(\phi^{cd})}{2}\sigma^2 \underbrace{-v^{cd}\int_{\phi^{cd}}^{\phi}\rho_e'''(\varrho)\frac{(\phi-\varrho)^2}{2}d\varrho}_{I_0}. \qquad (2.11)$$

By virtue of (2.11), we then compute $I_{1,1}$, $I_{1,2}$ and $I_{1,3}$ as follows:

$$\begin{aligned}
I_{1,1} = &-\frac{1}{2}\partial_t\left(\frac{v^{cd}}{v^2}(\partial_x\sigma)^2\right) - \partial_t\left(\frac{\partial_x\sigma}{v}\sigma\partial_x\left(\frac{v^{cd}}{v}\right)\right) - \frac{1}{2}\partial_t\left(\frac{v^{cd}}{v^2}\right)(\partial_x\sigma)^2 \\
&+ \partial_x\left(\frac{v^{cd}}{v}\right)\frac{\partial_x\sigma}{v}\partial_t\sigma - v^{cd}\partial_x\left(\frac{\partial_x\sigma}{v}\right)\sigma\partial_t(v^{-1}) + \frac{1}{2}\partial_t\left(v^{cd}\rho_e'(\phi^{cd})\sigma^2\right) \\
&+ \frac{1}{3}\partial_t\left(v^{cd}\rho_e''(\phi^{cd})\sigma^3\right) - \partial_t I_0\sigma \underbrace{+\frac{\partial_t v}{v}v^{cd}\rho_e'(\phi^{cd})\sigma^2}_{I_3} + \frac{1}{2}\partial_t\left(v^{cd}\rho_e'(\phi^{cd})\right)\sigma^2 \\
&+ \frac{1}{2}\frac{\partial_t v}{v}v^{cd}\rho_e''(\phi^{cd})\sigma^3 + \frac{1}{6}\partial_t\left(v^{cd}\rho_e''(\phi^{cd})\right)\sigma^3 - \frac{\partial_t v}{v}I_0\sigma + \partial_t\left(v^{cd}\partial_x\left(\frac{\partial_x\phi^{cd}}{v}\right)\right)\sigma v^{-1} \\
&+ \partial_x\partial_t\left(\frac{v^{cd}\sigma\partial_x\sigma}{v^2}\right) - \partial_x\left(\frac{v^{cd}\partial_t\sigma\partial_x\sigma}{v^2}\right),
\end{aligned} \qquad (2.12)$$

$$\begin{aligned}
I_{1,2} = &\frac{v^{cd}}{v^3}\partial_x^2\sigma\partial_x\sigma\psi + \frac{v^{cd}}{v}\partial_x^2\sigma\sigma\partial_x(\psi v^{-2}) + v^{cd}\partial_x\sigma\partial_x(v^{-1})\partial_x(\sigma\psi v^{-2}) \\
&\underbrace{-v^{cd}\rho_e'(\phi^{cd})\frac{\partial_x\sigma\sigma\psi}{v} - v^{cd}\rho_e'(\phi^{cd})\frac{\partial_x v\sigma^2\psi}{v^2} - \partial_x\left(v^{cd}\rho_e'(\phi^{cd})\right)\frac{\sigma^2\psi}{v}}_{I_4} - v^{cd}\rho_e''(\phi^{cd})\frac{\partial_x\sigma\sigma^2\psi}{v} \\
&- \frac{1}{2}v^{cd}\rho_e''(\phi^{cd})\frac{\partial_x v\sigma^3\psi}{v^2} - \frac{1}{2}\partial_x\left(v^{cd}\rho_e''(\phi^{cd})\right)\frac{\sigma^3\psi}{v} + \frac{\partial_x I_0\sigma\psi}{v} + \frac{\partial_x v I_0\sigma\psi}{v^2} \\
&+ v^{cd}\partial_x\left(\frac{\partial_x\phi^{cd}}{v}\right)\partial_x\left(\psi\sigma v^{-2}\right) - \partial_x\left(v^{cd}\partial_x\left(\frac{\partial_x\phi}{v}\right)\psi\sigma v^{-2}\right),
\end{aligned} \qquad (2.13)$$

$$\begin{aligned}
I_{1,3} = &-\partial_x\left(\frac{\partial_x\sigma}{v}\right)\frac{\psi\partial_x v^{cd}}{v(v^{cd})^2\rho_e'(\phi^{cd})}\underbrace{-\frac{\partial_x v^{cd}\sigma\psi}{(v^{cd})^2}}_{I_5}\underbrace{-\frac{1}{2}\frac{\rho_e''(\phi^{cd})\partial_x v^{cd}\sigma^2\psi}{(v^{cd})^2\rho_e'(\phi^{cd})}}_{I_6} \\
&+ \frac{\partial_x v^r I_0\psi}{(v^{cd})^3\rho_e'(\phi^{cd})} - \partial_x\left(\frac{\partial_x\phi^{cd}}{v}\right)\frac{\psi\partial_x v^{cd}}{v(v^{cd})^2\rho_e'(\phi^{cd})}.
\end{aligned} \qquad (2.14)$$

Note that I_l $(2 \le l \le 6)$ can not be directly controlled. To overcome this difficulty, we first get from (2.4) and (2.11) that

$$\begin{aligned}
(I_2 + I_5) + I_6 + I_4 = &-\frac{\partial_x v^{cd}\sigma\psi\varphi(v+v^{cd})}{(vv^{cd})^2} - \frac{\rho_e''(\phi^{cd})\partial_x v^{cd}\sigma^2\psi}{2\rho_e'(\phi^{cd})(v^{cd})^2} - v^{cd}\rho_e'(\phi^{cd})\frac{\partial_x\sigma\sigma\psi}{v} \\
&- \frac{\partial_x v\sigma^2\psi}{v^2}v^{cd}\rho_e'(\phi^{cd}) - \left(v^{cd}\rho_e'(\phi^{cd})\right)_x\frac{\sigma^2\psi}{v} \\
=&\frac{\partial_x v^{cd}\sigma^2\psi}{(v^{cd})^2}\left[v^{cd}\rho_e'(\phi^{cd}) - \frac{\rho_e''(\phi^{cd})}{2\rho_e'(\phi^{cd})}\right] - \frac{\partial_x\sigma\sigma\psi}{v}v^{cd}\rho_e'(\phi^{cd}) + \frac{\partial_x v^{cd}\sigma^2\psi}{v}\frac{\rho_e''(\phi^{cd})}{v^{cd}\rho_e'(\phi^{cd})} \\
&- \frac{\partial_x v\sigma^2\psi}{v^2}v^{cd}\rho_e'(\phi^{cd}) + \frac{1}{2}\frac{\partial_x v^{cd}\sigma^3\psi(v+v^{cd})}{vv^{cd}}\rho_e''(\phi^{cd}) - \frac{\partial_x v^{cd}\sigma\psi I_0(v+v^{cd})}{v(v^{cd})^2} \\
&+ \frac{\partial_x v^{cd}\sigma\psi\partial_x\left(\frac{\partial_x\phi^{cd}}{v}\right)(v+v^{cd})}{v^2v^{cd}} + \frac{\partial_x v^{cd}\sigma\psi\partial_x\left(\frac{\partial_x\sigma}{v}\right)(v+v^{cd})}{v^2v^{cd}},
\end{aligned}$$

which is further equal to

$$
\underbrace{\frac{1}{2}\frac{\partial_x\psi\sigma^2}{v}v^{cd}\rho'_e(\phi^{cd})}_{I_7}+\sigma^2\psi v^{cd}\rho'_e(\phi^{cd})\partial_x\left(\frac{1}{v}-\frac{1}{v^{cd}}\right)+\frac{1}{2}\frac{\sigma^2\psi\varphi}{v^2}\partial_x v^{cd}\left[\rho'_e(\phi^{cd})-\frac{v\rho_e(\phi^{cd})\rho''_e(\phi^{cd})}{v^{cd}\rho'_e(\phi^{cd})}\right]
$$

$$
-\frac{1}{2}\frac{\sigma^2\psi\partial_x\varphi}{v^2}v^{cd}\rho'_e(\phi^{cd})+\frac{1}{2}\frac{\partial_x v^{cd}\sigma^3\psi(v+v^{cd})}{vv^{cd}}\rho''_e(\phi^{cd})-\frac{\partial_x v^{cd}\sigma\psi I_0(v+v^{cd})}{v(v^{cd})^2} \tag{2.15}
$$

$$
+\frac{\partial_x v^{cd}\sigma\psi\partial_x\left(\frac{\partial_x\phi^{cd}}{v}\right)(v+v^{cd})}{v^2v^{cd}}+\frac{\partial_x v^{cd}\sigma\psi\partial_x\left(\frac{\partial_x\sigma}{v}\right)(v+v^{cd})}{v^2v^{cd}}-\partial_x\left(\frac{\psi\sigma^2}{2v}v^{cd}\rho'_e(\phi^{cd})\right).
$$

For I_3 and I_7, it follows from $(1.7)_1$, (2.1) and (2.4) that

$$
\begin{aligned}
I_3+I_7 =&\frac{3\partial_t\varphi\sigma^2}{2v}v^{cd}\rho'_e(\phi^{cd})+\frac{\partial_x u^{cd}\sigma^2}{v}v^{cd}\rho'_e(\phi^{cd})\\
=&\partial_t\left(\frac{3\varphi\sigma^2}{2v}v^{cd}\rho'_e(\phi^{cd})\right)-\frac{3}{2}v^{cd}\rho'_e(\phi^{cd})\varphi\partial_t\left(\frac{\sigma^2}{v}\right)\\
&-\frac{3}{2}\partial_t\left(v^{cd}\rho'_e(\phi^{cd})\right)\frac{\varphi\sigma^2}{v}+\frac{\partial_x u^{cd}\sigma^2}{v}v^{cd}\rho'_e(\phi^{cd}).
\end{aligned} \tag{2.16}
$$

Plugging (2.16), (2.15), (2.14), (2.13), (2.12) and (2.10) into (2.9), integrating the resulting identity with respect to x over $\mathbb{R}_+$, and using $(\mathcal{A}_2)$, we thus arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}_+}\left(\frac{1}{2}\psi^2+R\theta^{cd}\Phi\left(\frac{v}{v^{cd}}\right)+\frac{R}{\gamma-1}\theta^{cd}\Phi\left(\frac{\theta}{\theta^{cd}}\right)+\frac{v^{cd}}{2}|\rho'_e(\phi^{cd})|\sigma^2+\frac{v^{cd}}{2v^2}(\partial_x\sigma)^2\right)dx\\
&+\frac{d}{dt}\int_{\mathbb{R}_+}\frac{\partial_x\sigma}{v}\sigma\partial_x\left(\frac{v^{cd}}{v}\right)dx-\frac{3}{2}\frac{d}{dt}\int_{\mathbb{R}_+}v^{cd}\rho'_e(\phi^{cd})\frac{\varphi\sigma^2}{v}dx-\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}_+}v^{cd}\rho''_e(\phi^{cd})\sigma^3dx\\
&+\mu\int_{\mathbb{R}_+}\frac{(\partial_x\psi)^2}{v}dx+\int_{\mathbb{R}_+}\frac{\kappa}{v\theta}(\partial_x\zeta)^2dx\\
=&-\int_{\mathbb{R}_+}Q_1dx-\int_{\mathbb{R}_+}Q_2dx+\int_{\mathbb{R}_+}F\psi dx+\int_{\mathbb{R}_+}\frac{\zeta}{\theta}Gdx+\widetilde{H}(0,t)+\sum_{l=1}^{31}\mathcal{I}_l,
\end{aligned} \tag{2.17}
$$

where

$$
\begin{aligned}
\widetilde{H}=\widetilde{H}(x,t)=&(p-p^{cd})\psi-\mu\frac{\psi\partial_x\psi}{v}-\kappa\frac{\zeta}{\theta}\left(\frac{\partial_x\theta}{v}-\frac{\partial_x\theta^{cd}}{v^{cd}}\right)-\frac{\sigma\psi}{v}-\partial_t\left(\frac{v^{cd}\sigma\partial_x\sigma}{v^2}\right)\\
&+\frac{v^{cd}\partial_t\sigma\partial_x\sigma}{v^2}+v^{cd}\partial_x\left(\frac{\partial_x\phi}{v}\right)\psi\sigma v^{-2}+\frac{\psi\sigma^2}{2v}v^{cd}\rho'_e(\phi^{cd}),
\end{aligned}
$$

and

$$
\begin{cases}
\mathcal{I}_1 = \int_{\mathbb{R}_+} \partial_x \left(\frac{v^{cd}}{v} \right) \frac{\partial_x \sigma}{v} \partial_t \sigma dx, \quad \mathcal{I}_2 = -\int_{\mathbb{R}_+} v^{cd} \partial_x \left(\frac{\partial_x \sigma}{v} \right) \sigma \partial_t (v^{-1}) dx, \\[2mm]
\mathcal{I}_3 = -\int_{\mathbb{R}_+} \partial_t I_0 \sigma dx, \quad \mathcal{I}_4 = -\frac{1}{2} \int_{\mathbb{R}_+} \partial_t \left(\frac{v^{cd}}{v^2} \right) (\partial_x \sigma)^2 dx, \\[2mm]
\mathcal{I}_5 = -\int_{\mathbb{R}_+} \frac{\partial_t v}{v} I_0 \sigma dx, \quad \mathcal{I}_6 = \int_{\mathbb{R}_+} \partial_t \left(v^{cd} \partial_x \left(\frac{\partial_x \phi^{cd}}{v} \right) \right) \sigma v^{-1} dx, \\[2mm]
\mathcal{I}_7 = \int_{\mathbb{R}_+} \frac{v^{cd}}{v^3} \partial_x^2 \sigma \partial_x \sigma \psi dx, \quad \mathcal{I}_8 = \int_{\mathbb{R}_+} \frac{v^{cd}}{v} \partial_x^2 \sigma \sigma \partial_x (\psi v^{-2}) dx, \\[2mm]
\mathcal{I}_9 = \int_{\mathbb{R}_+} v^{cd} \partial_x \sigma \partial_x (v^{-1}) \partial_x (\sigma \psi v^{-2}) dx, \quad \mathcal{I}_{10} = \frac{1}{2} \int_{\mathbb{R}_+} \partial_t (v^{cd} \rho_e'(\phi^{cd})) \sigma^2 dx, \\[2mm]
\mathcal{I}_{11} = \frac{1}{2} \int_{\mathbb{R}_+} \frac{\partial_t v}{v} v^{cd} \rho_e''(\phi^{cd}) \sigma^3 dx, \quad \mathcal{I}_{12} = \int_{\mathbb{R}_+} \frac{\partial_x v^{cd} I_0 \psi}{(v^{cd})^3 \rho_e'(\phi^{cd})} dx, \\[2mm]
\mathcal{I}_{13} = \int_{\mathbb{R}_+} \frac{\partial_x v I_0 \sigma \psi}{v^2} dx, \quad \mathcal{I}_{14} = \int_{\mathbb{R}_+} \frac{\partial_x I_0 \sigma \psi}{v} dx, \\[2mm]
\mathcal{I}_{15} = -\int_{\mathbb{R}_+} \partial_x \left(\frac{\partial_x \sigma}{v} \right) \frac{\psi \partial_x v^{cd}}{v (v^{cd})^2 \rho_e'(\phi^{cd})} dx, \quad \mathcal{I}_{16} = \int_{\mathbb{R}_+} v^{cd} \partial_x \left(\frac{\partial_x \phi^{cd}}{v} \right) \partial_x \left(\psi \sigma v^{-2} \right) dx,
\end{cases}
$$

$$
\begin{cases}
\mathcal{I}_{17} = -\int_{\mathbb{R}_+} \partial_x \left(\frac{\partial_x \phi^{cd}}{v} \right) \frac{\psi \partial_x v^{cd}}{v (v^{cd})^2 \rho_e'(\phi^{cd})} dx, \quad \mathcal{I}_{18} = \frac{1}{6} \int_{\mathbb{R}_+} \partial_t \left(v^{cd} \rho_e''(\phi^{cd}) \right) \sigma^3 dx, \\[2mm]
\mathcal{I}_{19} = -\frac{1}{2} \int_{\mathbb{R}_+} v^{cd} \rho_e''(\phi^{cd}) \frac{\partial_x v \sigma^3 \psi}{v^2} dx, \quad \mathcal{I}_{20} = -\frac{1}{2} \int_{\mathbb{R}_+} \partial_x \left(v^{cd} \rho_e''(\phi^{cd}) \right) \frac{\sigma^3 \psi}{v} dx, \\[2mm]
\mathcal{I}_{21} = -\int_{\mathbb{R}_+} v^{cd} \rho_e''(\phi^{cd}) \frac{\partial_x \sigma \sigma^2 \psi}{v} dx, \quad \mathcal{I}_{22} = \int_{\mathbb{R}_+} \sigma^2 \psi v^{cd} \rho_e'(\phi^{cd}) \partial_x \left(\frac{1}{v} - \frac{1}{v^{cd}} \right) dx, \\[2mm]
\mathcal{I}_{23} = \frac{1}{2} \int_{\mathbb{R}_+} \frac{\sigma^2 \psi \varphi}{v^2} \partial_x v^{cd} \left[\rho_e'(\phi^{cd}) - \frac{v \rho_e(\phi^{cd}) \rho_e''(\phi^{cd})}{v^{cd} \rho_e'(\phi^{cd})} \right] dx, \\[2mm]
\mathcal{I}_{24} = -\frac{1}{2} \int_{\mathbb{R}_+} \frac{\sigma^2 \psi \partial_x \varphi}{v^2} v^{cd} \rho_e'(\phi^{cd}) dx, \quad \mathcal{I}_{25} = \frac{1}{2} \int_{\mathbb{R}_+} \frac{\partial_x v^{cd} \sigma^3 \psi (v + v^{cd})}{v v^{cd}} \rho_e''(\phi^{cd}) dx, \\[2mm]
\mathcal{I}_{26} = -\int_{\mathbb{R}_+} \frac{\partial_x v^{cd} \sigma \psi I_0 (v + v^{cd})}{v (v^{cd})^2} dx, \quad \mathcal{I}_{27} = \int_{\mathbb{R}_+} \frac{\partial_x v^{cd} \sigma \psi \partial_x \left(\frac{\partial_x \phi^{cd}}{v} \right) (v + v^{cd})}{v^2 v^{cd}} dx, \\[2mm]
\mathcal{I}_{28} = \int_{\mathbb{R}_+} \frac{\partial_x v^{cd} \sigma \psi \partial_x \left(\frac{\partial_x \sigma}{v} \right) (v + v^{cd})}{v^2 v^{cd}} dx, \quad \mathcal{I}_{29} = \int_{\mathbb{R}_+} \frac{\partial_x u^{cd} \sigma^2}{v} v^{cd} \rho_e'(\phi^{cd}) dx, \\[2mm]
\mathcal{I}_{30} = -\frac{3}{2} \int_{\mathbb{R}_+} \partial_t \left(v^{cd} \rho_e'(\phi^{cd}) \right) \frac{\varphi \sigma^2}{v} dx, \quad \mathcal{I}_{31} = -\frac{3}{2} \int_{\mathbb{R}_+} v^{cd} \rho_e'(\phi^{cd}) \varphi \partial_t \left(\frac{\sigma^2}{v} \right) dx.
\end{cases}
$$

We now turn to estimate the right hand side of (2.17) term by term. It should be noted that the following Poincaré type inequalities play an important role in our computations:

$$
|\zeta(x,t)| \le x^{\frac{1}{2}} \|\partial_x \zeta\|, \quad |\varphi(x,t)| \le |\varphi(0,t)| + x^{\frac{1}{2}} \|\partial_x \varphi\|, \quad |\sigma(x,t)| \le x^{\frac{1}{2}} \|\partial_x \sigma\|. \tag{2.18}
$$

From (2.18) and Lemma 4.1, one can further obtain

$$
\begin{cases}
\int_{\mathbb{R}_+} \varphi^2 \left((\partial_x \theta^{cd})^2 + |\partial_x^2 \theta^{cd}| \right) dx \le C\delta^2 \|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu} t} + C\delta^2 \|\partial_x \varphi\|^2, \\[3mm]
\int_{\mathbb{R}_+} (\zeta^2 + \sigma^2) \left((\partial_x \theta^{cd})^2 + |\partial_x^2 \theta^{cd}| \right) dx \le C\delta^2 \|\partial_x [\zeta, \sigma]\|^2,
\end{cases} \tag{2.19}
$$

where the following Sobolev inequality is also used:

$$
|h(x)| \le \sqrt{2} \|h\|^{1/2} \|\partial_x h\|^{1/2} \text{ for } h(x) \in H^1(\mathbb{R}_+). \tag{2.20}
$$

By applying (2.19), Lemma 4.2, the *a priori* assumption (2.7), Cauchy-Schwarz's inequality with $0 < \eta < 1$ and Sobolev's inequality (2.20), we obtain the estimates for terms involving Q_1 and Q_2 as follows:

$$\left| \int_{\mathbb{R}_+} Q_1 dx \right| \leq C \int_{\mathbb{R}} (\varphi^2 + \zeta^2) \left((\partial_x \theta^{cd})^2 + |\partial_x^2 \theta^{cd}| \right) dx \leq C\delta \|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu} t} + C\delta \|\partial_x [\varphi, \zeta]\|^2, \qquad (2.21)$$

$$\left| \int_{\mathbb{R}_+} Q_2 dx \right| \leq (C\epsilon_0 + \eta) \|\partial_x \zeta\|^2 + C_\eta \int_{\mathbb{R}_+} (\varphi^2 + \zeta^2)(\partial_x \theta^{cd})^2 dx$$
$$\leq (C\epsilon_0 + \eta)\|\partial_x \zeta\|^2 + C_\eta \delta^2 \|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu} t} + C_\eta \delta^2 \|\partial_x [\varphi, \zeta]\|^2. \qquad (2.22)$$

For the terms involving F and G, noticing that

$$|\partial_t u^{cd}| = O(1)\delta(1+t)^{-\frac{3}{2}} e^{-\frac{c_1 x^2}{1+t}}, \ \ |\partial_x^2 u^{cd}| = |\partial_x \partial_t v^{cd}| = O(1)\delta(1+t)^{-\frac{3}{2}} e^{-\frac{c_1 x^2}{1+t}}, \ \text{as } x \to +\infty,$$

we get from Cauchy-Schwarz's inequality that

$$\left| \int_{\mathbb{R}_+} F\psi dx \right| \leq \int_{\mathbb{R}_+} |\partial_t u^{cd} \psi| dx + C \int_{\mathbb{R}_+} |\partial_x^2 u^{cd} \psi| \, dx + C \int_{\mathbb{R}_+} |\partial_x u^{cd} \partial_x v^{cd} \psi| \, dx + C \int_{\mathbb{R}_+} |\partial_x u^{cd} \partial_x \varphi \psi| \, dx$$
$$\leq C\delta(1+t)^{-1-\alpha} \|\psi\|^2 + C\delta(1+t)^{-3/2+\alpha} + C\delta\|\partial_x \varphi\|^2, \qquad (2.23)$$

where $0 < \alpha < 1/2$, and

$$\left| \int_{\mathbb{R}_+} \frac{\zeta}{\theta} G dx \right| \leq C\|\zeta\|_\infty \|\partial_x u\|^2 \leq C\epsilon_0 \|\partial_x \psi\|^2 + C\epsilon_0 (1+t)^{-\frac{3}{2}}. \qquad (2.24)$$

We next compute the term $\widetilde{H}(0,t)$ arising from the boundary. Since $\zeta(0,t) = \sigma(0,t) = 0$, $\left| \widetilde{H}(0,t) \right|$ can be reduced to

$$\left| \frac{R\theta_- \varphi(0,t)}{v(0,t)v_-} \psi(0,t) + \mu \left(\frac{\psi \partial_x \psi}{v} \right) (0,t) \right|,$$

which is further dominated by

$$C|\varphi(0,t)\psi(0,t)| + C|(\partial_t \varphi)(0,t)\psi(0,t)|$$
$$\leq C|\varphi_0(0)|e^{-\frac{p_-}{\mu} t} |\psi(0,t)| \leq C\|\varphi_0(x)\|_{H^1} \|\psi\|^{1/2} \|\partial_x \psi\|^{1/2} e^{-\frac{p_-}{\mu} t} \qquad (2.25)$$
$$\leq C\|\varphi_0(x)\|_{H^1}^{4/3} e^{-\frac{p_-}{\mu} t} + C\epsilon_0^2 \|\partial_x \psi\|^2,$$

according to Lemma 4.2, Sobolev's inequality (2.20) and Young's inequaity.

In order to estimate $\mathcal{I}_l$ $(1 \leq l \leq 31)$, we first calculate

$$I_0 \sim \sigma^3, \ \ \partial_t I_0 = -v^{cd} \partial_t \phi \int_{\phi^{cd}}^{\phi} (\phi - \varrho) \rho_e'''(\varrho) d\varrho + \frac{1}{2}\sigma^2 \partial_t \phi^{cd} v^{cd} \rho_e'''(\phi^{cd}) - \partial_t v^{cd} \int_{\phi^{cd}}^{\phi} \frac{(\phi - \varrho)^2}{2} \rho_e'''(\varrho) d\varrho$$
$$\sim \partial_t \phi \sigma^2 + \partial_t v^{cd} \sigma^2 + \partial_t v^{cd} \sigma^3 = \partial_t \sigma \sigma^2 + 2\partial_t v^{cd} \sigma^2 + \partial_t v^{cd} \sigma^3, \quad (2.26)$$

and similarly,

$$\partial_x I_0 \sim \partial_x \sigma \sigma^2 + 2\partial_x v^{cd} \sigma^2 + \partial_x v^{cd} \sigma^3. \qquad (2.27)$$

In addition, from (2.4) and (2.5), it follows

$$\|\partial_t \sigma\|^2 + \|\partial_t \partial_x \sigma\|^2 \leq C\|\partial_x \psi\|^2 + C\epsilon_0 \left\| \left[\partial_x \varphi, \partial_x^2 \psi, \partial_x \sigma, \partial_x^2 \sigma \right] \right\|^2 + C\delta(1+t)^{-\frac{3}{2}}. \qquad (2.28)$$

For the sake of completeness, the proof of (2.28) is given in the appendix.

With (2.26), (2.27) and (2.28) in hand, we now employ (2.19), Cauchy-Schwarz's inequality with $0 < \eta < 1$, Sobolev's inequality and Lemma 4.1 repeatedly to present the following estimates:

$$|\mathcal{I}_1| \leq C \int_{\mathbb{R}_+} \left| \partial_x[v^{cd}, v] \partial_x \sigma \partial_t \sigma \right| dx \leq C \epsilon_0 \left\| [\partial_x \sigma, \partial_x^2 \sigma, \partial_t \sigma] \right\|^2,$$

$$|\mathcal{I}_2| \leq C \int_{\mathbb{R}_+} \left| \partial_x^2 \sigma \sigma \partial_x[\psi, u^{cd}] \right| dx + C \int_{\mathbb{R}_+} \left| \partial_x \sigma \sigma \partial_x[\varphi, v^{cd}] \partial_x[\psi, u^{cd}] \right| dx \leq C \epsilon_0 \left\| \partial_x [\varphi, \psi, \sigma, \partial_x \sigma] \right\|^2,$$

$$|\mathcal{I}_3| + |\mathcal{I}_5| + |\mathcal{I}_{11}| + |\mathcal{I}_{18}|$$
$$\leq C \int_{\mathbb{R}_+} \left| \partial_x u^{cd} \sigma^3 \right| dx + C \int_{\mathbb{R}_+} \left| \sigma^3[\partial_t \sigma, \partial_x \psi] \right| dx + C \int_{\mathbb{R}_+} \left| \partial_x \psi \sigma^4 \right| dx + C \int_{\mathbb{R}} \left| \partial_x u^{cd} \sigma^4 \right| dx$$
$$\leq C \epsilon_0 \left\| [\partial_t \sigma, \partial_x \sigma, \partial_x \psi] \right\|^2,$$

$$|\mathcal{I}_4| \leq C \int_{\mathbb{R}_+} \left| \partial_t[v^{cd}, v] \right| |\partial_x \sigma|^2 dx \leq C \epsilon_0 \left\| \partial_x[\sigma, \partial_x \sigma] \right\|^2,$$

$$|\mathcal{I}_6| \leq \int_{\mathbb{R}_+} \left| \partial_t \partial_x v^{cd} \left(\frac{\partial_x \phi^{cd}}{v} \right) \sigma v^{-1} \right| dx + \int_{\mathbb{R}_+} \left| \partial_t v^{cd} \left(\frac{\partial_x \phi^{cd}}{v} \right) \partial_x \sigma v^{-1} \right| dx$$
$$+ \int_{\mathbb{R}_+} \left| \partial_t v^{cd} \left(\frac{\partial_x \phi^{cd}}{v} \right) \sigma \partial_x(v^{-1}) \right| dx + \int_{\mathbb{R}_+} \left| \partial_x v^{cd} \partial_t \left(\frac{\partial_x \phi^{cd}}{v} \right) \sigma v^{-1} \right| dx$$
$$+ \int_{\mathbb{R}_+} \left| v^{cd} \partial_t \left(\frac{\partial_x \phi^{cd}}{v} \right) \partial_x \sigma v^{-1} \right| dx + \int_{\mathbb{R}_+} \left| v^{cd} \partial_t \left(\frac{\partial_x \phi^{cd}}{v} \right) \sigma \partial_x(v^{-1}) \right| dx$$
$$\leq C \delta(1+t)^{-\frac{3}{2}} + C \epsilon_0 \| \partial_x[\varphi, \psi, \partial_x \sigma] \|^2,$$

$$|\mathcal{I}_7| \leq C \|\psi\|_{L^\infty} (\|\partial_x \sigma\|^2 + \|\partial_x^2 \sigma\|^2) \leq C \epsilon_0 \|\partial_x[\sigma, \partial_x \sigma]\|^2,$$

$$|\mathcal{I}_8| \leq C \int_{\mathbb{R}_+} |\partial_x^2 \sigma \sigma \partial_x \psi| dx + C \int_{\mathbb{R}_+} |\partial_x^2 \sigma \sigma \psi \partial_x \varphi| dx + C \int_{\mathbb{R}_+} |\partial_x^2 \sigma \sigma \psi \partial_x v^{cd}| dx$$
$$\leq C \epsilon_0 \|\partial_x[\psi, \varphi, \partial_x \sigma]\|^2 + C \epsilon_0 \int_{\mathbb{R}_+} (\partial_x \theta^{cd})^2 \sigma^2 dx \leq C \epsilon_0 \|\partial_x[\psi, \varphi, \sigma, \partial_x \sigma]\|^2,$$

$$|\mathcal{I}_9| \leq C \int_{\mathbb{R}_+} |\partial_x \sigma \partial_x v \partial_x \sigma \psi| dx + C \int_{\mathbb{R}_+} |\partial_x \sigma \partial_x v \sigma \partial_x \psi| dx + C \int_{\mathbb{R}_+} |\partial_x \sigma \partial_x v \sigma \psi \partial_x v| dx$$
$$\leq C \|\partial_x \sigma\|_{H^1} \|\psi\|_{H^1} \|\partial_x \sigma\| \|\partial_x v\| + C \|\partial_x \sigma\|_{H^1} \|\sigma\|_{H^1} \|\partial_x \psi\| \|\partial_x v\| + C \|\partial_x \sigma\|_{H^1} \|\sigma\|_{H^1} \|\psi\|_{H^1} \|\partial_x \varphi\|^2$$
$$+ C \|\partial_x \sigma\| \|\partial_x \varphi\| \|\sigma\|_{H^1} \|\psi\|_{H^1} \|\partial_x v^{cd}\|_{H^1} + C \|\psi\|_{H^1} \left(\|\partial_x \sigma\|^2 + \|\sigma(\partial_x v^{cd})^2\|^2 \right)$$
$$\leq C \epsilon_0 \|\partial_x[\sigma, \partial_x \sigma, \varphi, \psi]\|^2,$$

$$|\mathcal{I}_{10}| + |\mathcal{I}_{29}| + |\mathcal{I}_{30}| \leq C \int_{\mathbb{R}_+} \left| \partial_x u^{cd} \sigma^2 \right| dx \leq C \int_{\mathbb{R}_+} \left((\partial_x \theta^{cd})^2 + |\partial_x^2 \theta^{cd}| \right) \sigma^2 dx \leq C \delta \|\partial_x \sigma\|^2,$$

$$|\mathcal{I}_{12}| + |\mathcal{I}_{13}| + |\mathcal{I}_{14}| + |\mathcal{I}_{19}| + |\mathcal{I}_{20}| + |\mathcal{I}_{25}| + |\mathcal{I}_{26}|$$
$$\leq C \int_{\mathbb{R}_+} \left| \partial_x v^{cd} \sigma^3 \psi \right| dx + C \int_{\mathbb{R}_+} \left| \partial_x[\varphi, \sigma] \sigma^3 \psi \right| dx$$
$$\leq C \epsilon_0 \|\partial_x[\sigma, \varphi]\|^2 + C \epsilon_0 \int_{\mathbb{R}_+} (\partial_x \theta^{cd})^2 \sigma^2 dx \leq C \epsilon_0 \|\partial_x[\sigma, \varphi]\|^2,$$

$$|\mathcal{I}_{15}| \leq \eta \int_{\mathbb{R}_+} \left| \partial_x^2 \sigma \right|^2 dx + C_\eta \int_{\mathbb{R}_+} \psi^2 (\partial_x \theta^{cd})^2 dx + C \int_{\mathbb{R}_+} \left| \partial_x \sigma \partial_x \varphi \right|^2 dx + C \int_{\mathbb{R}_+} \left| \partial_x \sigma \partial_x v^{cd} \right|^2 dx$$
$$\leq (C \epsilon_0 + \eta) \|\partial_x[\sigma, \partial_x \sigma]\|^2 + C_\eta \int_{\mathbb{R}_+} \psi^2 (\partial_x \theta^{cd})^2 dx,$$

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$$|\mathcal{I}_{16}| \leq C \int_{\mathbb{R}_+} |\partial_x^2 v^{cd} \partial_x \sigma \psi|\, dx + C \int_{\mathbb{R}_+} |\partial_x^2 v^{cd} \sigma \partial_x \psi|\, dx$$

$$+ C \int_{\mathbb{R}_+} |\partial_x^2 v^{cd}[\partial_x \varphi, \partial_x v^{cd}]\sigma\psi|\, dx + C \int_{\mathbb{R}_+} |\partial_x v^{cd}[\partial_x \varphi, \partial_x v^{cd}]\partial_x \sigma\psi|\, dx$$

$$+ C \int_{\mathbb{R}_+} |\partial_x v^{cd}[\partial_x \varphi, \partial_x v^{cd}]\sigma\partial_x \psi|\, dx + C \int_{\mathbb{R}_+} |\partial_x v^{cd}[\partial_x \varphi, \partial_x v^{cd}]^2\sigma\psi|\, dx$$

$$\leq C\epsilon_0 \|\partial_x[\sigma, \varphi, \psi]\|^2 + C\delta(1+t)^{-\frac{3}{2}},$$

$$|\mathcal{I}_{17}| + |\mathcal{I}_{27}| \leq C \int_{\mathbb{R}_+} |(\partial_x v^{cd})^2 \partial_x v\psi|\, dx + C \int_{\mathbb{R}_+} |\partial_x^2 v^{cd}\partial_x v^{cd}\psi|\, dx$$

$$\leq C\epsilon_0 \|\partial_x \varphi\|^2 + C \int_{\mathbb{R}_+} \psi^2 (\partial_x \theta^{cd})^2 dx + C\delta^2(1+t)^{-\frac{3}{2}},$$

$$|\mathcal{I}_{21}| + |\mathcal{I}_{22}| + |\mathcal{I}_{23}| + |\mathcal{I}_{24}| \leq C \int_{\mathbb{R}_+} |\partial_x[\sigma, \varphi]\sigma^2\psi|\, dx + C \int_{\mathbb{R}_+} |\sigma^2\psi\varphi\partial_x v^{cd}|\, dx$$

$$\leq C\epsilon_0 \|\partial_x[\sigma, \varphi]\|^2 + C\epsilon_0 \int_{\mathbb{R}_+} (\partial_x \theta^{cd})^2 \varphi^2 dx$$

$$\leq C\epsilon_0 \|\partial_x[\sigma, \varphi]\|^2 + C\delta^2 \|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu}t},$$

$$|\mathcal{I}_{28}| \leq C \int_{\mathbb{R}_+} |\partial_x^2 \sigma \partial_x v^{cd}\sigma\psi|\, dx + C \int_{\mathbb{R}_+} |\partial_x \sigma\partial_x v\partial_x v^{cd}\psi\sigma|\, dx$$

$$\leq C\epsilon_0 \|\partial_x[\sigma, \partial_x \sigma, \varphi]\|^2 + C\epsilon_0 \int_{\mathbb{R}_+} (\partial_x \theta^{cd})^2\sigma^2 dx \leq C\epsilon_0 \|\partial_x[\sigma, \partial_x \sigma, \varphi]\|^2.$$

For the last term $\mathcal{I}_{31}$, applying (2.11) again, one can see that

$$\mathcal{I}_{31} = -\frac{3}{2} \int_{\mathbb{R}_+} v^{cd}\rho_e'(\phi^{cd})\partial_t \left(\frac{\sigma^2}{v}\right)\left(-v^{cd}\partial_x \left(\frac{\partial_x \sigma}{v}\right) + v\left(1 - v^{cd}\rho_e(\sigma + \phi^{cd})\right) - v^{cd}\partial_x \left(\frac{\partial_x \phi^{cd}}{v}\right)\right) dx,$$

which implies

$$\left|\mathcal{I}_{31} - \frac{d}{dt} \int_{\mathbb{R}_+} \left(v^{cd}\rho_e'(\phi^{cd})\right)^2 \sigma^3 dx\right| \leq C\epsilon_0 \left\|\left[\partial_x \psi, \partial_x \varphi, \partial_t \sigma, \partial_x \sigma, \partial_x^2 \sigma\right]\right\|^2 + C\delta(1+t)^{-\frac{3}{2}}.$$

Let us now define $\sigma_0(x) = \sigma(x, 0) = \phi(x, 0) - \phi^{cd}(x, 0)$. From the Poisson equation (2.4), it follows that for any $t \geq 0$

$$\|\sigma(t)\|_{H^1}^2 \leq C\|\varphi(t)\|^2 + C\left\|\partial_x^2 \phi^{cd}(t)\right\|^2 + C\left\|\left(\partial_x \phi^{cd}\right)^2 (t)\right\|^2 \leq C\|\varphi(t)\|^2 + C\delta^2,$$

and hence in particular,

$$\|\sigma_0\|_{H^1}^2 \leq C\|\varphi_0\|^2 + C\delta^2. \tag{2.29}$$

We now conclude from (2.17), (2.21), (2.22), (2.23), (2.24), (2.25), (2.28), (2.29) and the above estimates on $\mathcal{I}_l$ $(1 \leq l \leq 31)$ that

$$\|[\psi, \varphi, \zeta]\|^2 + \|\sigma\|_{H^1}^2 + \epsilon_0 \|\partial_x \varphi\|^2 + \int_0^T \|\partial_x[\psi, \zeta]\|^2 dt$$

$$\leq C \|[\psi_0, \zeta_0]\|^2 + C \|\varphi_0\|_{H^1}^{4/3} + (C\epsilon_0 + \eta) \int_0^T \|\partial_x[\varphi, \partial_x \psi, \sigma, \partial_x \sigma]\|^2 dt \tag{2.30}$$

$$+ C_\eta \int_0^T \int_{\mathbb{R}_+} \psi^2 (\partial_x \theta^{cd})^2 dx dt + C\delta,$$

for suitably small $\epsilon_0 > 0$, $\delta > 0$ and $\eta > 0$.

Step 2. *Dissipation of $\partial_x[\varphi, \sigma, \partial_x \sigma]$.*

We first differentiate (2.1) with respect to x, to obtain

$$\partial_t \partial_x \varphi - \partial_x^2 \psi = 0. \tag{2.31}$$

Then multiplying (2.4), (2.2) and (2.31) by $\partial_x^2 \sigma$, $-v\partial_x \varphi$ and $\mu \partial_x \varphi$, respectively, and integrating the resulting equalities with respect to x over $\mathbb{R}_+$, one has

$$
\begin{aligned}
\int_{\mathbb{R}_+} & \frac{v^{cd}}{v}(\partial_x^2 \sigma)^2 dx + \int_{\mathbb{R}_+} vv^{cd}|\rho'_e(\phi^{cd})|(\partial_x \sigma)^2 dx \\
&= \int_{\mathbb{R}_+} \frac{v^{cd}}{v^2} \partial_x v \partial_x \sigma \partial_x^2 \sigma dx - \int_{\mathbb{R}_+} \partial_x v \left[1 - v^{cd}\rho_e(\sigma + \phi^{cd})\right] \partial_x \sigma dx + \int_{\mathbb{R}_+} v\partial_x [v^{cd}\rho'_e(\phi^{cd})]\sigma \partial_x \sigma dx \\
&\quad + \int_{\mathbb{R}_+} v\partial_x \left[\frac{v^{cd}\rho''_e(\phi^{cd})}{2}\sigma^2\right] \partial_x \sigma dx + \int_{\mathbb{R}_+} \partial_x \varphi \partial_x \sigma dx + \varphi(0,t)\partial_x \sigma(0,t) - \int_{\mathbb{R}_+} v\partial_x I_0 \partial_x \sigma dx \\
&\quad - \int_{\mathbb{R}_+} v^{cd} \partial_x \left(\frac{\partial_x \phi^{cd}}{v}\right) \partial_x^2 \sigma dx,
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}_+} & \partial_t \psi v \partial_x \varphi dx - \int_{\mathbb{R}_+} (\partial_x p - \partial_x p^{cd})v\partial_x \varphi dx + \int_{\mathbb{R}_+} \partial_x \varphi \partial_x \sigma dx + \int_{\mathbb{R}_+} \left(\frac{\partial_x \phi^{cd}}{v} - \frac{\partial_x \phi^{cd}}{v^{cd}}\right) v\partial_x \varphi dx \\
&= -\int_{\mathbb{R}_+} \mu \partial_x^2 \psi \partial_x \varphi dx - \int_{\mathbb{R}_+} \mu \partial_x^2 u^{cd} \partial_x \varphi dx - \mu \int_{\mathbb{R}_+} \partial_x (v^{-1}) \partial_x u v \partial_x \varphi dx + \int_{\mathbb{R}_+} \partial_t u^{cd} v \partial_x \varphi dx,
\end{aligned}
\tag{2.33}
$$

and

$$\int_{\mathbb{R}_+} \mu \left(\partial_t \partial_x \varphi - \partial_x^2 \psi\right) \partial_x \varphi dx = 0. \tag{2.34}$$

The summation of (2.32), (2.33) and (2.34) further implies

$$
\begin{aligned}
&-\frac{d}{dt}\int_{\mathbb{R}_+} \psi v \partial_x \varphi dx + \frac{\mu}{2}\frac{d}{dt}\int_{\mathbb{R}_+} (\partial_x \varphi)^2 dx + \int_{\mathbb{R}_+} p^{cd}(\partial_x \varphi)^2 dx \\
&\quad + \int_{\mathbb{R}_+} \frac{v^{cd}}{v}(\partial_x^2 \sigma)^2 dx + \int_{\mathbb{R}_+} vv^{cd}|\rho'_e(\phi^{cd})|(\partial_x \sigma)^2 dx \\
&= \underbrace{-\int_{\mathbb{R}_+} \psi \partial_t v \partial_x \varphi dx}_{J_1} \underbrace{-\int_{\mathbb{R}_+} \psi v \partial_t \partial_x \varphi dx}_{J_2} + \underbrace{\int_{\mathbb{R}_+} R\partial_x \left[\frac{\zeta}{v}\right] v\partial_x \varphi dx}_{J_3} - \underbrace{\int_{\mathbb{R}_+} R\varphi \partial_x \left[\frac{\theta^{cd}}{vv^{cd}}\right] v\partial_x \varphi dx}_{J_4} \\
&\quad \underbrace{-\int_{\mathbb{R}_+} \left(\frac{\partial_x \phi^{cd}}{v} - \frac{\partial_x \phi^{cd}}{v^{cd}}\right) v\partial_x \varphi dx}_{J_5} \underbrace{-\int_{\mathbb{R}_+} \mu \partial_x^2 u^{cd} \partial_x \varphi dx}_{J_6} \underbrace{-\mu \int_{\mathbb{R}_+} \partial_x (v^{-1}) \partial_x u v \partial_x \varphi dx}_{J_7} \\
&\quad \underbrace{+\int_{\mathbb{R}_+} \frac{v^{cd}}{v^2} \partial_x v \partial_x \sigma \partial_x^2 \sigma dx}_{J_8} \underbrace{-\int_{\mathbb{R}_+} \partial_x v \left[1 - v^{cd}\rho_e(\sigma + \phi^{cd})\right] \partial_x \sigma dx}_{J_9} \underbrace{+\varphi(0,t)\partial_x \sigma(0,t)}_{J_{10}} \\
&\quad \underbrace{-\int_{\mathbb{R}_+} v\partial_x I_0 \partial_x \sigma dx}_{J_{11}} + \underbrace{\int_{\mathbb{R}_+} v\partial_x \left[v^{cd}\rho'_e(\phi^{cd})\right]\sigma \partial_x \sigma dx}_{J_{12}} + \underbrace{\int_{\mathbb{R}_+} v\partial_x \left[\frac{v^{cd}\rho''_e(\phi^{cd})}{2}\sigma^2\right] \partial_x \sigma dx}_{J_{13}} \\
&\quad \underbrace{+\int_{\mathbb{R}_+} \partial_t u^{cd} v \partial_x \varphi dx}_{J_{14}} \underbrace{-\int_{\mathbb{R}_+} v^{cd} \partial_x \left(\frac{\partial_x \phi^{cd}}{v}\right) \partial_x^2 \sigma dx}_{J_{15}}.
\end{aligned}
\tag{2.35}
$$

We now turn to compute J_l ($1 \leq l \leq 15$) term by term. For brevity, we directly give the following computations:

$$|J_1| \leq C\int_{\mathbb{R}_+} |\psi \partial_x \psi \partial_x \varphi|\, dx + C\int_{\mathbb{R}_+} |\psi \partial_x u^{cd} \partial_x \varphi|\, dx \leq C\epsilon_0 \|\partial_x[\psi, \varphi]\|^2 + C\delta(1+t)^{-3/2},$$

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$$|J_2| \leq |\psi(0,t)v(0,t)(\partial_t\varphi)(0,t)| + \left|\int_{\mathbb{R}_+} v(\partial_x\psi)^2 dx\right| + \left|\int_{\mathbb{R}_+} \psi\partial_x v\partial_x\psi dx\right|$$

$$\leq C\|\varphi_0\|_{H^1}^{\frac{4}{3}}e^{-\frac{p_-}{\mu}t} + C\epsilon_0\|\partial_x[\psi,\varphi]\|^2 + C\|\partial_x\psi\|^2 + C\int_{\mathbb{R}_+}\psi^2(\partial_x\theta^{cd})^2 dx,$$

$$|J_3| + |J_4| + |J_5| \leq (\eta + C\epsilon_0)\|\partial_x\varphi\|^2 + C_\eta\|\partial_x\zeta\|^2 + C_\eta\int_{\mathbb{R}_+}(\varphi^2 + \zeta^2)(\partial_x\theta^{cd})^2 dx$$

$$\leq (\eta + C\epsilon_0)\|\partial_x[\varphi,\zeta]\|^2 + C_\eta\|\partial_x\zeta\|^2 + C\delta\|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu}t},$$

$$|J_6| \leq C\delta\|\partial_x\varphi\|^2 + C\delta(1+t)^{-5/2},$$

$$|J_7| \leq C\int_{\mathbb{R}_+}(\partial_x\varphi)^2|\partial_x\psi|dx + C\int_{\mathbb{R}_+}(\partial_x\varphi)^2|\partial_x u^{cd}|dx + C\int_{\mathbb{R}_+}|\partial_x v^{cd}\partial_x u^{cd}\partial_x\varphi|dx + \int_{\mathbb{R}_+}|\partial_x v^{cd}\partial_x\psi\partial_x\varphi|dx$$

$$\leq C\epsilon_0\|\partial_x[\psi,\partial_x\psi,\varphi]\|^2 + C\delta(1+t)^{-2},$$

$$|J_8| + |J_9| \leq C\epsilon_0\|\partial_x[\varphi,\partial_x\sigma,\sigma]\|^2,$$

$$|J_{10}| \leq C\epsilon_0\|\partial_x\sigma\|_{H^1}^2 + C\|\varphi_0\|_{H^1}^{\frac{4}{3}}e^{-\frac{4p_-}{3\mu}t},$$

$$|J_{11}| + |J_{12}| + |J_{13}| + |J_{14}| + |J_{15}| \leq C\epsilon_0\|\partial_x[\varphi,\sigma,\partial_x\sigma]\|^2 + C\delta(1+t)^{-3/2}.$$

Substituting the above estimations for J_l ($1 \leq l \leq 15$) into (2.35), letting $\eta > 0$ be suitably small and combing (2.30), we obtain

$$\|[\psi,\varphi,\zeta]\|^2 + \|\sigma\|_{H^1}^2 + \|\partial_x\varphi\|^2 + \int_0^T \|\partial_x\sigma\|_{H^1}^2 dt + \int_0^T \|\partial_x[\varphi,\psi,\zeta]\|^2 dt$$

$$\leq C\epsilon_0\int_0^T \|\partial_x^2\psi\|^2 dt + C\delta + C\|[\psi_0,\zeta_0]\|^2 + C\|\varphi_0\|_{H^1}^{4/3} + C\int_0^T\int_{\mathbb{R}_+}\psi^2(\partial_x\theta^{cd})^2 dxdt. \tag{2.36}$$

Step 3. *Higher order energy estimates.*
Multiplying (2.2) by $-\partial_x^2\psi$, and integrating the resultant equality with respect to x over $\mathbb{R}_+$, one has

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+}(\partial_x\psi)^2 dx + \mu\int_{\mathbb{R}_+}\frac{(\partial_x^2\psi)^2}{v}dx$$

$$= \underbrace{-\int_{\mathbb{R}_+}\left(\frac{\partial_x\phi}{v} - \frac{\partial_x\phi^{cd}}{v^{cd}}\right)\partial_x^2\psi dx}_{J_{16}} + \underbrace{\int_{\mathbb{R}_+}\partial_x(p - p^{cd})\partial_x^2\psi dx}_{J_{17}} + \underbrace{\mu\int_{\mathbb{R}_+}\frac{\partial_x\psi\partial_x\varphi}{v^2}\partial_x^2\psi dx}_{J_{18}}$$

$$\underbrace{+\mu\int_{\mathbb{R}_+}\frac{\partial_x\psi\partial_x v^{cd}}{v^2}\partial_x^2\psi dx}_{J_{19}} \underbrace{-\int_{\mathbb{R}_+}F\partial_x^2\psi dx}_{J_{20}} \underbrace{-(\partial_t\psi\partial_t\varphi)(0,t)}_{J_{21}}. \tag{2.37}$$

To obtain the estimates for J_l ($16 \leq l \leq 21$), we use Cauchy-Schwarz's inequality with $0 < \eta < 1$, Sobolev's inequlity (2.20) and (2.19) repeatedly to perform the calculations as follows:

$$|J_{16}| \leq C\int_{\mathbb{R}_+}\left|\partial_x v^{cd}\varphi\partial_x^2\psi\right|dx + C\int_{\mathbb{R}_+}\left|\partial_x\sigma\partial_x^2\psi\right|dx$$

$$\leq (C\delta + \eta)\|\partial_x[\varphi,\partial_x\psi]\|^2 + C_\eta\|\partial_x\sigma\|^2 + C\delta\|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu}t},$$

$$|J_{17}| \leq C\int_{\mathbb{R}_+}\left|\partial_x[\zeta,\varphi]\partial_x^2\psi\right|dx + C\int_{\mathbb{R}_+}\left|[\zeta,\varphi]\partial_x[\varphi,v^{cd}]\partial_x^2\psi\right|dx$$

$$\leq (C\epsilon_0 + \eta)\|\partial_x[\varphi,\partial_x\psi]\|^2 + C_\eta\|\partial_x[\zeta,\varphi]\|^2 + C\delta\|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu}t},$$

$$|J_{18}| + |J_{19}| \leq C\epsilon_0\|\partial_x[\psi,\partial_x\psi]\|^2,$$

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$$|J_{20}| \leq C \int_{\mathbb{R}_+} |\partial_x^2 u^{cd} \partial_x^2 \psi|\, dx + C \int_{\mathbb{R}_+} |\partial_x u^{cd} \partial_x v \partial_x^2 \psi|\, dx + C \int_{\mathbb{R}_+} |\partial_t u^{cd} \partial_x^2 \psi|\, dx$$

$$\leq C\delta \|\partial_x [\varphi, \partial_x \psi]\|^2 + C\delta (1+t)^{-\frac{5}{2}}.$$

For the last term J_{21}, in light of Lemma 4.2, we have

$$J_{21} = -(\partial_t \psi \partial_t \varphi)(0,t) = -\partial_t [(\psi \partial_t \varphi)(0,t)] + \varphi_0(0) \frac{(p_-)^2}{\mu^2} \psi(0,t) e^{-\frac{p_-}{\mu} t}, \tag{2.38}$$

furthermore, it follows that

$$|(\psi \partial_t \varphi)(0,T)| \leq C \varphi_0(0) \psi(0,T) e^{-\frac{p_-}{\mu} T} \leq C \epsilon_0 \|\varphi_0\|_{H^1} e^{-\frac{p_-}{\mu} T}, \tag{2.39}$$

and

$$|(\psi \partial_t \varphi)(0,0)| \leq C |\psi_0(0) \varphi_0(0)| \leq C(\|\psi_0\|_{H^1}^2 + \|\varphi_0\|_{H^1}^2). \tag{2.40}$$

By virtue of (2.38), (2.39) and (2.40) and carrying out the similar calculations as (2.25), we thereby obtain

$$\left| \int_0^T J_{21} dt \right| \leq C \epsilon_0 \int_0^T \|\partial_x \psi\|^2 dt + C \|\varphi_0\|_{H^1}^{\frac{4}{3}} + C \|\psi_0\|_{H^1}^2 + C \epsilon_0 \|\varphi_0\|_{H^1}.$$

Plug the above estimations for J_l ($16 \leq l \leq 21$) into (2.37), and recall (2.36) and (2.30), then choose $\epsilon_0 > 0$, $\delta > 0$ and $\eta > 0$ suitably small, to derive

$$\|[\psi, \varphi, \zeta]\|^2 + \|\sigma\|_{H^1}^2 + \|\partial_x \varphi\|^2 + \|\partial_x \psi\|^2$$
$$+ \int_0^T \|\partial_x \sigma\|_{H^1}^2 dt + \int_0^T \|\partial_x [\varphi, \psi, \zeta]\|^2 dt + \int_0^T \|\partial_x^2 \psi\|^2 dt \tag{2.41}$$
$$\leq C\delta + C \|\zeta_0\|^2 + C \|[\psi_0, \varphi_0]\|_{H^1}^{4/3} + C \int_0^T \int_{\mathbb{R}_+} \psi^2 (\partial_x \theta^{cd})^2 dx dt.$$

Similarly, multiplying (2.3) by $-\partial_x^2 \zeta$, and integrating the resulting equality over $\mathbb{R}_+$, we obtain

$$\frac{R}{2(\gamma-1)} \frac{d}{dt} \int_{\mathbb{R}_+} (\partial_x \zeta)^2 dx + \kappa \int_{\mathbb{R}_+} \frac{(\partial_x^2 \zeta)^2}{v} dx$$
$$= \int_{\mathbb{R}_+} (p \partial_x u - p^{cd} \partial_x u^{cd}) \partial_x^2 \zeta dx + \kappa \int_{\mathbb{R}_+} \frac{\partial_x \zeta \partial_x \varphi}{v^2} \partial_x^2 \zeta dx + \kappa \int_{\mathbb{R}_+} \frac{\partial_x \zeta \partial_x v^{cd}}{v^2} \partial_x^2 \zeta dx \tag{2.42}$$
$$+ \kappa \int_{\mathbb{R}_+} \partial_x \left(\frac{\varphi \partial_x \theta^{cd}}{v v^{cd}} \right) \partial_x^2 \zeta dx - \int_{\mathbb{R}_+} G \partial_x^2 \zeta dx,$$

where we have used boundary condition $\zeta(0,t) = 0$. The right hand side of (2.42) can be handled as J_l ($16 \leq l \leq 21$), the details of which we omit, therefore one can get from (2.42) and (2.41) that

$$\|[\psi, \varphi, \zeta]\|_{H^1}^2 + \|\sigma\|_{H^1}^2 + \int_0^T \|\partial_x \sigma\|_{H^1}^2 dt + \int_0^T \|\partial_x [\varphi, \psi, \zeta]\|^2 dt + \int_0^T \|\partial_x^2 [\psi, \zeta]\|^2 dt$$
$$\leq C\delta + C \|[\varphi_0, \psi_0, \zeta_0]\|_{H^1}^{4/3} + C \int_0^T \int_{\mathbb{R}_+} \psi^2 (\partial_x \theta^{cd})^2 dx dt. \tag{2.43}$$

Finally, letting $\delta > 0$ small enough, combing (2.43) and (4.3) in Lemma 4.3, we obtain (2.8) as desired, this completes the proof of Proposition 2.1.

$\square$

3 Global existence and large time behavior

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. In view of the energy estimates obtained in Proposition 2.1, one sees that

$$\sup_{0 \le t \le T} \|[\varphi, \psi, \zeta, \sigma](t)\|_{H^1}^2 \le C\delta + C \|[\psi_0, \zeta_0, \varphi_0]\|_{H^1}^{4/3}. \tag{3.1}$$

Notice that $\delta > 0$ is a parameter independent of ϵ_0. By letting $\delta > 0$ be small enough, the global existence of the solution of the Cauchy problem (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) then follows from the standard continuation argument based on the local existence (cf. [24]) and the *a priori* estimate (2.8). Moreover, (3.1) implies (1.22). Our intention next is to prove the large time behavior as (1.23). For this, we first justify the following limits:

$$\lim_{t \to +\infty} \|\partial_x[\varphi, \psi, \zeta](t)\|_{L^2}^2 = 0, \tag{3.2}$$

and

$$\lim_{t \to +\infty} \|\partial_x\sigma(t)\|^2 = 0. \tag{3.3}$$

To prove (3.2) and (3.3), we get from (2.1), (2.2), (2.3), (2.8) and (1.20) that

$$\int_0^{+\infty} \left| \frac{d}{dt} \|\partial_x[\varphi, \psi, \zeta]\|^2 \right| dt = 2 \int_0^{+\infty} |(\partial_t\partial_x[\varphi, \psi, \zeta], \partial_x[\varphi, \psi, \zeta])| \, dt$$
$$\le C + C \int_0^{+\infty} \|\partial_x [\varphi, \psi, \zeta, \sigma, \partial_x [\psi, \zeta, \sigma]]\|^2 \, dt < +\infty. \tag{3.4}$$

On that other hand, (2.28), (2.36) and (2.8) yield

$$\int_0^{+\infty} \left| \frac{d}{dt} \|\partial_x\sigma\|^2 \right| dt = 2 \int_0^{+\infty} |(\partial_t\partial_x\sigma, \partial_x\sigma)| \, dt < +\infty. \tag{3.5}$$

Consequently, (3.4), (3.5) together with (2.8) gives (3.2) and (3.3). Then (1.23) follows from (3.2), (3.3) and Sobolev's inequality (2.20). This ends the proof of Theorem 1.1.

□

4 Appendix

In this appendix, we will give some basic results used in the paper. The first lemma is borrowed from [24].

Lemma 4.1. *Let θ^{cd} satisfy (1.18), for $|\theta_+ - \theta_-| = \delta$, it holds that*

$$\int_{\mathbb{R}_+} (\partial_x\theta^{cd})^4 dx \le C\delta^4(1+t)^{-\frac{3}{2}}, \quad \int_{\mathbb{R}_+} (\partial_x^2\theta^{cd})^2 dx \le C\delta^2(1+t)^{-\frac{3}{2}},$$

$$\int_{\mathbb{R}_+} (\partial_x^3\theta^{cd})^2 dx \le C\delta^2(1+t)^{-\frac{5}{2}}, \quad \int_{\mathbb{R}_+} x\left((\partial_x\theta^{cd})^2 + |\partial_x^2\theta^{cd}|\right) dx \le C\delta.$$

Next is the key observation from the boundary condition (2.5).

Lemma 4.2. *It holds that*

$$\varphi(0, t) = \varphi_0(0)e^{-\frac{p_-}{\mu}t}. \tag{4.1}$$

Proof. Since $\partial_x u^{cd}(0, t) = 0$, from (2.5) it follows that

$$\frac{R\theta_-}{v_- + \varphi(0, t)} - \mu\frac{\partial_t\varphi(0, t)}{v_- + \varphi(0, t)} = p_-, \quad t > 0,$$

which implies

$$\partial_t \varphi(0,t) = -\frac{p_-}{\mu} \varphi(0,t). \tag{4.2}$$

(4.1) follows from (4.2) and the compatibility condition $\varphi(0,0) = \varphi_0(0)$. This ends the proof of Lemma 4.2.

$\square$

We now give the following estimates concerning the delicate term $\int_0^T \int_{\mathbb{R}_+} (\partial_x \theta^{cd})^2 \psi^2 dx dt$.

Lemma 4.3. *Assume all the conditions listed in Proposition 2.1 hold, then for any $0 \leq T \leq +\infty$, there exists an energy functional $\mathcal{E}(\varphi, \psi, \zeta)$ with*

$$|\mathcal{E}(\varphi, \psi, \zeta)| \leq C\delta^2 \|[\varphi, \psi, \zeta]\|^2,$$

such that the following energy estimate holds

$$\mathcal{E}(\varphi, \psi, \zeta)(T) + \int_0^T \int_{\mathbb{R}_+} (\partial_x \theta^{cd})^2 \psi^2 dx dt \leq C\delta + C\delta \|\varphi_0\|_{H^1}^2 + C\delta \int_0^T \|\partial_x[\varphi, \psi, \zeta, \sigma, \partial_x \sigma]\|^2 dt. \tag{4.3}$$

Proof. Define

$$w = \int_0^x (\partial_y \theta^{cd})^2 dy.$$

It is easy to check that

$$\|w(\cdot,t)\|_\infty \leq C\delta^2 (1+t)^{-\frac{1}{2}}, \quad \|\partial_t w(\cdot,t)\|_\infty \leq C\delta^2 (1+t)^{-\frac{3}{2}}. \tag{4.4}$$

From (2.4) and (2.11), it follows that

$$\sigma = -\frac{\varphi}{vv^{cd}\rho_e'(\phi^{cd})} - \frac{1}{vv^{cd}\rho_e'(\phi^{cd})} \underbrace{\left[v^{cd}\partial_x \left(\frac{\partial_x \sigma}{v}\right) + \left(\frac{v^{cd}\rho_e''(\phi^{cd})}{2}\sigma^2 - I_0\right)v + v^{cd}\partial_x \left(\frac{\partial_x \phi^{cd}}{v}\right)\right]}_{\mathcal{M}}. \tag{4.5}$$

On the other hand, (2.2) can be rewritten as

$$\partial_t \psi + \partial_x \left(\frac{R\zeta - p^{cd}\varphi - \sigma}{v}\right) = -\partial_x \left(\frac{1}{v}\right)\sigma - \frac{\varphi\partial_x\phi^{cd}}{vv^{cd}} + \mu\partial_x\left(\frac{\partial_x\psi}{v}\right) + F. \tag{4.6}$$

Substituting (4.5) into (4.6), one has

$$\partial_t \psi + \partial_x \left(\frac{R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi}{v}\right) = \partial_x \left(\frac{\mathcal{M}}{v}\right) - \partial_x \left(\frac{1}{v}\right)\sigma - \frac{\varphi\partial_x\phi^{cd}}{vv^{cd}} + \mu\partial_x\left(\frac{\partial_x\psi}{v}\right) + F. \tag{4.7}$$

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Multiplying (4.7) by $\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]vw$, integrating the resulting equation over $\mathbb{R}_+$ leads to

$$\frac{1}{2}\int_{\mathbb{R}_+}\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]^2(\partial_x\theta^{cd})^2dx$$

$$=\frac{d}{dt}\int_{\mathbb{R}_+}\psi\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]vwdx - \underbrace{\int_{\mathbb{R}_+}\psi\partial_t\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]vwdx}_{\mathcal{K}_1}$$

$$-\underbrace{\int_{\mathbb{R}_+}\psi\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]\partial_tvwdx}_{\mathcal{K}_2} - \underbrace{\int_{\mathbb{R}_+}\psi\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]v\partial_twdx}_{\mathcal{K}_3}$$

$$-\underbrace{\int_{\mathbb{R}_+}\frac{\partial_xv}{v}\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]^2wdx}_{\mathcal{K}_4} + \underbrace{\mu\int_{\mathbb{R}_+}\frac{\partial_x\psi}{v}\partial_x\left[\left(R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right)vw\right]dx}_{\mathcal{K}_5}$$

$$-\underbrace{\int_{\mathbb{R}_+}F\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]vwdx}_{\mathcal{K}_6} - \underbrace{\int_{\mathbb{R}_+}\frac{\partial_x\varphi}{v^2}\sigma\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]vwdx}_{\mathcal{K}_7}$$

$$-\underbrace{\int_{\mathbb{R}_+}\frac{\partial_xv^{cd}}{v^2}\sigma\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]vwdx}_{\mathcal{K}_8} + \underbrace{\int_{\mathbb{R}_+}\frac{\mathcal{M}}{v}\partial_x\left[\left(R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right)vw\right]dx}_{\mathcal{K}_9}$$

$$+\underbrace{\int_{\mathbb{R}_+}\frac{\varphi\partial_x\phi^{cd}}{vv^{cd}}\left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd}\right)\varphi\right]vwdx}_{\mathcal{K}_{10}}.$$

$$(4.8)$$

We now turn to compute $\mathcal{K}_l$ $(1 \le l \le 10)$ term by term. For the delicate term $\mathcal{K}_1$, it can be rewritten as

$$\mathcal{K}_1 = -\int_{\mathbb{R}_+}\psi\partial_t\left(R\zeta - p^{cd}\varphi\right)vwdx - \int_{\mathbb{R}_+}\psi\partial_t\left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})}\varphi\right)vwdx$$

$$= -(\gamma-1)\int_{\mathbb{R}_+}\psi vw\partial_t\left(\frac{R}{\gamma-1}\zeta + p^{cd}\varphi\right)dx + \gamma\int_{\mathbb{R}_+}\psi vwp^{cd}\partial_x\psi dx$$

$$+\gamma\int_{\mathbb{R}_+}\psi vw\partial_tp^{cd}\varphi dx - \frac{1}{2}\int_{\mathbb{R}_+}\partial_x(\psi^2)\left(\frac{w}{v^{cd}\rho_e'(\phi^{cd})}\right)dx - \int_{\mathbb{R}_+}\psi\varphi\partial_t\left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})}\right)vwdx$$

$$= \underbrace{(\gamma-1)\int_{\mathbb{R}_+}\psi w\left(R\zeta - p^{cd}\varphi\right)\left(\partial_xu^{cd} + \partial_x\psi\right)dx}_{\mathcal{K}_{1,1}} + \underbrace{\kappa(\gamma-1)\int_{\mathbb{R}_+}\frac{v^{cd}\partial_x\zeta - \partial_x\theta^{cd}\varphi}{vv^{cd}}\partial_x\left(\psi vw\right)dx}_{\mathcal{K}_{1,2}}$$

$$(4.9)$$

$$\underbrace{-(\gamma-1)\int_{\mathbb{R}_+}\psi vwGdx}_{\mathcal{K}_{1,3}} \underbrace{-\frac{\gamma}{2}\int_{\mathbb{R}_+}p^{cd}\partial_xvw\psi^2dx}_{\mathcal{K}_{1,4}} \underbrace{-\frac{\gamma}{2}\int_{\mathbb{R}_+}\partial_xp^{cd}vw\psi^2dx}_{\mathcal{K}_{1,5}} \underbrace{+\gamma\int_{\mathbb{R}_+}\psi vw\partial_tp^{cd}\varphi dx}_{\mathcal{K}_{1,6}}$$

$$+\int_{\mathbb{R}_+}\left(\frac{1}{2v^{cd}\rho_e'(\phi^{cd})} - \frac{\gamma}{2}p^{cd}v\right)\psi^2(\partial_x\theta^{cd})^2dx + \underbrace{\frac{1}{2}\int_{\mathbb{R}_+}\psi^2\partial_x\left(\frac{1}{v^{cd}\rho_e'(\phi^{cd})}\right)wdx}_{\mathcal{K}_{1,7}}$$

$$\underbrace{-\int_{\mathbb{R}_+}\psi\varphi\partial_t\left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})}\right)vwdx}_{\mathcal{K}_{1,8}},$$

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where in the third identity we have used

$$\frac{R}{\gamma-1}\partial_t\zeta + p^{cd}\partial_t\varphi = -\frac{R\zeta - p^{cd}\varphi}{v}\left(\partial_x u^{cd} + \partial_x\psi\right) + \kappa\partial_x\left(\frac{v^{cd}\partial_x\zeta - \partial_x\theta^{cd}\varphi}{vv^{cd}}\right) + G,$$

which is derived from (2.1) and (2.3).

Since $\rho'_e(\phi^{cd}) < 0$ according to the assumption $(\mathcal{A})_2$, (4.9) further implies

$$0 < -\int_{\mathbb{R}_+}\left(\frac{1}{2v^{cd}\rho'_e(\phi^{cd})} - \frac{\gamma}{2}p^{cd}v\right)\psi^2(\partial_x\theta^{cd})^2 dx = -\mathcal{K}_1 + \sum_{l=1}^{8}\mathcal{K}_{1,l}. \tag{4.10}$$

To compute $\mathcal{K}_{1,l}$ $(1 \le l \le 8)$ and $\mathcal{K}_l$ $(2 \le l \le 10)$, by applying (2.19), (4.4), Cauchy-Schwarz's inequality, Sobolev's inequality (2.20), Young's inequality and Lemmas 4.2 and 4.1, we directly address the following estimates:

$$\begin{aligned}
|\mathcal{K}_{1,1}| + |\mathcal{K}_{1,6}| + |\mathcal{K}_{1,8}| + |\mathcal{K}_2| \le &C\int_{\mathbb{R}_+}|\psi w|(|\partial_x u^{cd}| + |\partial_t\theta^{cd}|)(|\zeta| + |\varphi|)dx + C\int_{\mathbb{R}_+}|\psi w\partial_x\psi|(|\zeta| + |\varphi|)dx \\
\le &C\|w\partial_t\theta^{cd}\|_{L^\infty}\|[\varphi,\psi,\zeta]\|^2 + C\delta\|\partial_x\psi\|^2 + \frac{C}{\delta}\int_{\mathbb{R}_+}w^2\psi^2[\varphi,\zeta]^2 dx \\
\le &C\delta\|\partial_x\psi\|^2 + C\delta\epsilon_0(1+t)^{-\frac{3}{2}} + \frac{C}{\delta^2}\|w\|_{L^\infty}^4\|\psi\|^2\|[\varphi,\zeta]\|^4 \\
\le &C\delta\|\partial_x\psi\|^2 + C\delta\epsilon_0(1+t)^{-\frac{3}{2}},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{K}_{1,2}| + |\mathcal{K}_5| \le &C\int_{\mathbb{R}_+}\left(|\partial_x\varphi| + |\partial_x\zeta| + |\partial_x\theta^{cd}\varphi|\right)\left(|\partial_x\psi w| + |\psi(\partial_x\theta^{cd})^2| + |\psi w\partial_x v|\right)dx \\
\le &C\delta\|\partial_x[\varphi,\psi,\zeta]\|^2 + C\delta(1+t)^{-\frac{3}{2}},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{K}_{1,3}| \le &C\int_{\mathbb{R}_+}\left|\psi w\left((\partial_x u^{cd})^2 + (\partial_x\psi)^2\right)\right| dx \le C\int_{\mathbb{R}_+}\psi^2 w^2(\partial_x u^{cd})^2 dx + C\int_{\mathbb{R}_+}(\partial_x u^{cd})^2 dx + C\delta\|\partial_x\psi\|^2 \\
\le &C\delta\|\partial_x\psi\|^2 + C\delta(1+t)^{-\frac{3}{2}},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{K}_{1,4}| + |\mathcal{K}_{1,5}| + |\mathcal{K}_{1,7}| \le &C\int_{\mathbb{R}_+}\left|\partial_x\theta^{cd}w\psi^2\right| dx + C\int_{\mathbb{R}_+}\left|\partial_x\varphi w\psi^2\right| dx \\
\le &C\delta\int_{\mathbb{R}_+}(\partial_x\theta^{cd})^2\psi^2 dx + C\delta\|\partial_x[\psi,\varphi]\|^2 + C\delta(1+t)^{-2},
\end{aligned}$$

$$|\mathcal{K}_3| \le C\int_{\mathbb{R}_+}|\psi(|\zeta| + |\varphi|)\partial_t w| dx \le C\delta\epsilon_0(1+t)^{-\frac{3}{2}},$$

$$\begin{aligned}
|\mathcal{K}_4| + |\mathcal{K}_{10}| \le &C\int_{\mathbb{R}_+}\left|\partial_x\theta^{cd}w\left(\zeta^2 + \varphi^2\right)\right| dx + C\int_{\mathbb{R}_+}\left|\partial_x\varphi w\left(\zeta^2 + \varphi^2\right)\right| dx \\
\le &C\delta(1+t)^{-1/2}\int_{\mathbb{R}_+}|\partial_x\theta^{cd}||x|\left(\|\partial_x\zeta\|^2 + \|\partial_x\varphi\|^2 + |\varphi(0,t)|^2\right)dx \\
&+ C\delta\|\partial_x\varphi,\zeta\|^2 + \frac{C}{\delta}\|w\|_{L^\infty}^2\left(\|\zeta\|\|\partial_x\zeta\| + \|\varphi\|\|\partial_x\varphi\|\right)\|[\varphi,\zeta]\|^2 \\
\le &C\delta\|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu}t} + C\delta\|\partial_x[\varphi,\zeta]\|^2 + C\delta(1+t)^{-2},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{K}_6| \le &C\delta^2(1+t)^{-\frac{1}{2}}\|F\|_{L^1}\left(\|\zeta\|_\infty + \|\varphi\|_\infty\right) \le C\delta^2(1+t)^{-1}\left(\|\zeta\|^{\frac{1}{2}}\|\partial_x\zeta\|^{\frac{1}{2}} + \|\varphi\|^{\frac{1}{2}}\|\partial_x\varphi\|^{\frac{1}{2}}\right) \\
\le &C\delta\|\partial_x[\varphi,\zeta]\|^2 + C\delta(1+t)^{-\frac{4}{3}},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{K}_7| \le &C\int_{\mathbb{R}_+}|\partial_x\varphi\sigma(|\zeta| + |\varphi|)w| dx \le C\delta(1+t)^{-\frac{1}{2}}\left(\|\zeta\|_\infty + \|\varphi\|_\infty\right)\|\partial_x\varphi\|\|\sigma\| \\
\le &C\delta(1+t)^{-\frac{1}{2}}\left(\|\zeta\|^{\frac{1}{2}}\|\partial_x\zeta\|^{\frac{1}{2}} + \|\varphi\|^{\frac{1}{2}}\|\partial_x\varphi\|^{\frac{1}{2}}\right)\|\partial_x\varphi\|\|\sigma\| \\
\le &C\delta\|\partial_x[\varphi,\zeta]\|^2 + C\delta(1+t)^{-2},
\end{aligned}$$

$$|\mathcal{K}_8| \le C \int_{\mathbb{R}_+} |\partial_x \theta^{cd} w| |\sigma| (|\zeta| + |\varphi|) \, dx \le C\delta \|\varphi_0\|_{H^1}^2 e^{-\frac{p_-}{\mu}t} + C\delta \|\partial_x[\varphi, \zeta, \sigma]\|^2,$$

$$|\mathcal{K}_9| \le C \int_{\mathbb{R}_+} \left[|\partial_x^2 \sigma| + |\partial_x \sigma \partial_x v| + |\sigma^2| + |\partial_x v^{cd} \partial_x v| \right] \left[(|\partial_x \zeta| + |\partial_x \varphi|)|w| + (|\zeta| + |\varphi|)((\partial_x \theta^{cd})^2 + |\partial_x v w|) \right] dx$$

$$\le C\delta \|\partial_x[\varphi, \zeta, \sigma, \partial_x \sigma]\|^2 + C\delta(1+t)^{-\frac{3}{2}}.$$

We now plug the above estimates for $\mathcal{K}_{1,l}$ with $1 \le l \le 8$ into (4.10) to obtain

$$\left| \mathcal{K}_1 - \int_{\mathbb{R}_+} \left(\frac{1}{2v^{cd}\rho_e'(\phi^{cd})} - \frac{\gamma}{2} p^{cd} v \right) \psi^2 (\partial_x \theta^{cd})^2 dx \right| \tag{4.11}$$

$$\le C\delta \|\partial_x[\varphi, \psi, \zeta]\|^2 + C\delta(1+t)^{-\frac{3}{2}} + C\delta \int_{\mathbb{R}_+} (\partial_x \theta^{cd})^2 \psi^2 dx.$$

Next by substituting the estimates for $\mathcal{K}_l$ $(2 \le l \le 10)$ and (4.11) into (4.8) and integrating the resulting equality with respect to time over $[0, T]$, one has

$$-\int_{\mathbb{R}_+} \psi \left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd} \right) \varphi \right] vw \, dx$$

$$+ \int_0^T \int_{\mathbb{R}_+} \left\{ \frac{1}{2} \left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd} \right) \varphi \right]^2 + \left(-\frac{1}{2v^{cd}\rho_e'(\phi^{cd})} + \frac{\gamma}{2} p^{cd} v - C\delta \right) \psi^2 \right\} (\partial_x \theta^{cd})^2 dx dt$$

$$\le C\delta + C\delta\epsilon_0 + C\delta \|\varphi_0\|_{H^1}^2 + C\delta \int_0^T \|\partial_x[\varphi, \psi, \zeta, \sigma, \partial_x \sigma]\|^2 dt,$$

$$\tag{4.12}$$

for suitably small $\delta > 0$ and $\epsilon_0 > 0$.

Let us now define

$$\mathcal{E}(\varphi, \psi, \zeta) = -\int_{\mathbb{R}_+} \psi \left[R\zeta + \left(\frac{1}{vv^{cd}\rho_e'(\phi^{cd})} - p^{cd} \right) \varphi \right] vw \, dx, \tag{4.13}$$

then (4.3) follows from (4.12) and (4.13), this ends the proof of Lemma 4.3. $\qquad \square$

Finally we give the detailed proof of (2.28).

Proof of (2.28). Taking the inner product of ∂_t(2.4) with $\partial_t \sigma$ with respect to x over $\mathbb{R}_+$, one has

$$\underbrace{\int_{\mathbb{R}_+} \partial_t \left(\frac{v^{cd}}{v} \right) \partial_x^2 \sigma \partial_t \sigma dx}_{\mathcal{J}_1} + \underbrace{\int_{\mathbb{R}_+} \frac{v^{cd}}{v} \partial_t \partial_x^2 \sigma \partial_t \sigma dx}_{\mathcal{J}_2} - \underbrace{\int_{\mathbb{R}_+} \partial_t \left(\frac{v^{cd}}{v^2} \partial_x v \right) \partial_x \sigma \partial_t \sigma dx}_{\mathcal{J}_3} - \underbrace{\int_{\mathbb{R}_+} \frac{v^{cd}}{v^2} \partial_x v \partial_t \partial_x \sigma \partial_t \sigma dx}_{\mathcal{J}_4}$$

$$= -\int_{\mathbb{R}_+} \partial_t \varphi \partial_t \sigma dx + \int_{\mathbb{R}_+} \partial_t v \left(1 - v^{cd} \rho_e(\sigma + \phi^{cd}) \right) \partial_t \sigma dx$$

$$+ \int_{\mathbb{R}_+} v \partial_t \left(1 - v^{cd} \rho_e(\sigma + \phi^{cd}) \right) \partial_t \sigma dx - \int_{\mathbb{R}_+} \partial_t \left(v^{cd} \partial_x \left(\frac{\partial_x \phi^{cd}}{v} \right) \right) \partial_t \sigma dx.$$

$$\tag{4.14}$$

We turn our attention first to $\mathcal{J}_l$ $(1 \le l \le 4)$ which can not be directly controlled. Since $\sigma(0, t) = \sigma(+\infty, t) = 0$, by integration by parts and using the cancellation, we find

$$\mathcal{J}_2 + \mathcal{J}_4 = -\int_{\mathbb{R}_+} \frac{v^{cd}}{v} \partial_t \partial_x \sigma \partial_t \partial_x \sigma dx - \int_{\mathbb{R}_+} \frac{\partial_x v^{cd}}{v} \partial_t \partial_x \sigma \partial_t \sigma dx, \tag{4.15}$$

and

$$\mathcal{J}_1 + \mathcal{J}_3 = \int_{\mathbb{R}_+} \frac{\partial_t v^{cd}}{v} \partial_x^2 \sigma \partial_t \sigma dx + \int_{\mathbb{R}_+} \frac{v^{cd}}{v^2} \partial_t v \partial_x \sigma \partial_x \partial_t \sigma dx - \int_{\mathbb{R}_+} \frac{\partial_t v^{cd}}{v^2} \partial_x v \partial_x \sigma \partial_t \sigma dx. \tag{4.16}$$

On the other hand, similar to (2.11), one has

$$1 - v^{cd}\rho_e(\sigma + \phi^{cd}) = -v^{cd}\rho_e'(\phi^{cd})\sigma \underbrace{-v^{cd}\int_{\phi^{cd}}^{\phi}\rho_e''(\varrho)(\phi - \varrho)d\varrho}_{\mathcal{J}_0}, \tag{4.17}$$

and moreover

$$\partial_t\mathcal{J}_0 \sim \partial_t\sigma\sigma + \partial_t v^{cd}\sigma + \partial_t v^{cd}\sigma^2. \tag{4.18}$$

Substituting (4.15), (4.16), (4.17) and (4.18) into (4.14) and applying (2.1) and (1.7), we deduce

$$\int_{\mathbb{R}_+}\frac{v^{cd}}{v}|\partial_t\partial_x\sigma|^2dx - \int_{\mathbb{R}_+}vv^{cd}\rho_e'(\phi^{cd})|\partial_t\sigma|^2dx$$

$$\leq \left|\int_{\mathbb{R}_+}\frac{\partial_x v^{cd}}{v}\partial_t\partial_x\sigma\partial_t\sigma dx\right| + \left|\int_{\mathbb{R}_+}\frac{\partial_t v^{cd}}{v}\partial_x^2\sigma\partial_t\sigma dx\right| + \left|\int_{\mathbb{R}_+}\frac{v^{cd}}{v^2}\partial_t v\partial_x\sigma\partial_x\partial_t\sigma dx\right|$$

$$+ \left|\int_{\mathbb{R}_+}\frac{\partial_t v^{cd}}{v^2}\partial_x v\partial_x\sigma\partial_t\sigma dx\right| + C\int_{\mathbb{R}_+}|\partial_x\psi\partial_t\sigma|dx + C\int_{\mathbb{R}_+}|\partial_x u\partial_t\sigma|dx$$

$$+ C\int_{\mathbb{R}_+}|\partial_t v^{cd}\sigma\partial_t\sigma|dx + C\int_{\mathbb{R}_+}|\partial_t\sigma\sigma\partial_t\sigma|dx + C\int_{\mathbb{R}_+}|\partial_t v^{cd}\sigma^2\partial_t\sigma|dx$$

$$+ C\int_{\mathbb{R}_+}|\partial_t\partial_x^2 v^{cd}\partial_t\sigma|dx + C\int_{\mathbb{R}_+}|\partial_t v^{cd}\partial_x^2 v^{cd}\partial_t\sigma|dx + C\int_{\mathbb{R}_+}|\partial_t v^{cd}\partial_x v^{cd}\partial_x v\partial_t\sigma|dx$$

$$+ C\int_{\mathbb{R}_+}|\partial_x v^{cd}\partial_x^2 u\partial_t\sigma|dx,$$

which yields (2.28), according to Cauchy-Schwarz's inequality, (2.19) and Lemma 4.1. This completes the proof of (2.28).

$\square$

References

[1] F. V. Atkinson and L. A. Peletier, Similarity solutions of the nonlinear diffusion equation, *Arch. Ration. Mech. Anal.* **54** (1974), 373–392.

[2] D. Chae, On the nonexistence of global weak solutions to the Navier-Stokes-Poisson equations in $\mathbb{R}^N$, *Comm. Partial Differential Equations* **35** (2010), no. 3, 535–557.

[3] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure.* Dover Publications, Inc., New York, N. Y. 1957.

[4] S. Chapman and T. G. Colwing, *The Mathematical Theory of Non-uniform Gases*, 3rd ed., Cambridge Math. Lib., Cambridge University Press, Cambridge, 1990.

[5] F. Chen, *Introduction to Plasma Physics and Controlled Fusion.* Second edition. Plenum Press, 1984.

[6] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics.* Third edition. Springer-Verlag, Berlin, 2010. xxxvi+708 pp.

[7] S.-J. Ding, H.-Y. Wen, L. Yao and C.-J. Zhu, Global solutions to one-dimensional compressible Navier-Stokes-Poisson equations with density-dependent viscosity, *J. Math. Phys.* **50** (2009), no. 2, 023101, 17 pp.

[8] D. Donatelli, Local and global existence for the coupled Navier-Stokes-Poisson problem, *Quart. Appl. Math.* **61** (2003), no. 2, 345–361.

[9] D. Donatelli and P. Marcati, A quasineutral type limit for the Navier-Stokes-Poisson system with large data, *Nonlinearity* **21** (2008), no. 1, 135–148.

[10] R.-J. Duan, Green's function and large time behavior of the Navier-Stokes-Maxwell system, *Anal. Appl. (Singap.)* **10** (2012), no. 2, 133–197.

[11] R.-J. Duan and S.-Q. Liu, Global stability of rarefaction waves of the Navier-Stokes-Poisson system, *J. Differential Equations* **258** (2015), no. 7, 2495–2530.

[12] R.-J. Duan and S.-Q. Liu, Global stability of rarefaction waves of the Vlasov-Poisson-Boltzmann system, arXiv:1405.2522.

[13] R.-J. Duan and X.-F. Yang, Stability of rarefaction wave and boundary layer for outflow problem on the two-fluid Navier-Stokes-Poisson equations, *Comm. Pure Appl. Anal.* **12** (2013), no. 2, 985-1014.

[14] B. Ducomet, A remark about global existence for the Navier-Stokes-Poisson system, *Appl. Math. Lett.* **12** (1999), no. 7, 31–37.

[15] S. Ghosh, S. Sarkar, M. Khan and M. R. GuptaIon, Ion acoustic shock waves in a collisional dusty plasma, *Physics of Plasmas* **378** (2002), no. 9, doi: 10.1063/1.1418429

[16] J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, *Arch. Ration. Mech. Anal.* **95** (1986), no. 4, 325–344.

[17] H. Grad, Asymptotic theory of the Boltzmann equation, *Physics of Fluids* **6** (1963), no. 2, 147–181.

[18] Y. Guo, The Vlasov-Poisson-Boltzmann system near Maxwellians, *Comm. Pure Appl. Math.* **55** (2002), 1104–1135.

[19] Y. Guo and J. Jang, Global Hilbert expansion for the Vlasov-Poisson-Boltzmann system, *Comm. Math. Phys.* **299** (2010), no. 2, 469–501.

[20] Y. Guo and B. Pausader, Global smooth ion dynamics in the Euler-Poisson system, *Comm. Math. Phys.* **303** (2011), 89–125.

[21] L. Hsiao and H.-L. Li, Compressible Navier-Stokes-Poisson equations, *Acta Math. Sci. Ser. B Engl. Ed.* **30** (2010), 1937–1948.

[22] L. Hsiao and T.-P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, *Comm. Math. Phys.* **143** (1992), no. 3, 599–605.

[23] F.-M. Huang, J. Li and A. Matsumura, Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system, *Arch. Ration. Mech. Anal.* **197** (2010), no. 1, 89–116.

[24] F.-M. Huang, A. Matsumura and X.-D. Shi, On the stability of contact discontinuity for compressible Navier-Stokes equations with free boundary, *Osaka J. Math.* **41** (2004), 193–210.

[25] F.-M. Huang, A. Matsumura and Z.-P. Xin, Stability of contact discontinuities for the 1-D compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.* **179** (2005), 55–77.

[26] F.-M. Huang, Y. Wang and X.-Y. Zhai, Stability of viscous contact wave for compressible Navier-Stokes system of general gas with free boundary, *Acta Math. Sci. Ser. B Engl.* **30** (2010), no.6, 1906–1916.

[27] F.-M. Huang, Z.-P. Xin and T. Yang, Contact discontinuity with general perturbations for gas motions, *Adv. Math.* **219** (2008), no. 4, 1246–1297.

[28] J. Jang and I. Tice, Instability theory of the Navier-Stokes-Poisson equations, *Anal. PDE* **6** (2013), no. 5, 1121–1181.

[29] Q.-S. Jiu, Y. Wang and Z.-P. Xin, Vacuum behaviors around rarefaction waves to 1D compressible Navier-Stokes equations with density-dependent viscosity, *SIAM J. Math. Anal.* **45** (2013), no. 5, 3194–3228.

[30] S. Kawashima and Y. Tanaka, Stability of rarefaction waves for a model system of a radiating gas, *Kyushu J. Math.* **58** (2004), no. 2, 211–250.

[31] S. Kawashima and P.-C. Zhu, Asymptotic stability of rarefaction wave for the Navier-Stokes equations for a compressible fluid in the half space, *Arch. Ration. Mech. Anal.* **194** (2009), no. 1, 105–132.

[32] S. K. Kundu, D. K. Ghosh, P. Chatterjee and B. Das, Shock waves in a dusty plasma with positive and negative dust, where electrons are superthermally distributed, *Bulg. J. Phys.* **38** (2011), 409–419.

[33] H.-L. Li, A. Matsumura and G.-J. Zhang, Optimal decay rate of the compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$, *Arch. Ration. Mech. Anal.* **196** (2010), no. 2, 681–713.

[34] T.-P. Liu and Z.-P. Xin, Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations, *Comm. Math. Phys.* **118** (1988), 451-465.

[35] T.-P. Liu, T. Yang, S.-H. Yu, and H.-J. Zhao, Nonlinear stability of rarefaction waves for the Boltzmann equation, *Arch. Ration. Mech. Anal.* **181** (2) (2006), 333–371.

[36] T.-P. Liu and S.-H. Yu, Boltzmann equation: micro-macro decompositions and positivity of shock profiles, *Comm. Math. Phys.* **246** (2004), 133–179.

[37] P. A. Markowich, C. A. Ringhofer and C. Schmeiser, *Semiconductor Equations*, Springer, New York, 1990.

[38] A. Matsumura, Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas, *Methods Appl. Anal.* **8** (2001), no. 4, 645–666.

[39] A. Matsumura and K. Nishihara, On the stability of travelling wave solutions of a one- dimensional model system for compressible viscous gas, *Japan J. Appl. Math.* **2** (1985), no. 1, 17–25.

[40] A. Matsumura and K. Nishihara, Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.* **3** (1986), 1–13.

[41] A. Matsumura and K. Nishihara, Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas, *Comm. Math. Phys.* **144** (1992), no. 2, 325-335.

[42] T. Pan, H.-X. Liu and K. Nishihara, Asymptotic behavior of a one-dimensional compressible viscous gas with free boundary, *SIAM J. Math. Anal.* **34** (2002), no. 2, 273–291.

[43] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, Berlin, 1983.

[44] Z. Tan, T. Yang, H.-J. Zhao and Q.-Y. Zou, Global solutions to the one-dimensional compressible Navier-Stokes-Poisson equations with large data, *SIAM J. Math. Anal.* **45** (2013), no. 2, 547–571.

[45] S.-H. Yu, Nonlinear wave propagations over a Boltzmann shock profile, *J. Amer. Math. Soc.* **23** (2010), no. 4, 1041–1118.

[46] G.-J Zhang, H.-L. Li and C.-J. Zhu, Optimal decay rate of the non-isentropic compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$, *J. Differential Equations* **250** (2011), no. 2, 866–891.

[47] T. Zhang and D.-Y. Fang, Global behavior of spherically symmetric Navier-Stokes-Poisson system with degenerate viscosity coefficients, *Arch. Ration. Mech. Anal.* **191** (2009), no. 2, 195–243.

[48] F. Zhou and Y.-P. Li, Convergence rate of solutions toward stationary solutions to the bipolar Navier-Stokes-Poisson equations in a half line, *Bound. Value Probl.* **124** (2013), 22 pp.